Recall Some Complexity Classes

Def: A set $A \subseteq \Sigma^\star$ is in DTIME[t(n)] iff there exists a deterministic, multi-tape TM, M, and a constant c, such that,

1. $A \quad = \quad \mathcal{L}(M) \quad \equiv \quad \{w \in \Sigma^\star \mid M(w) = 1\}$, and
2. $\forall w \in \Sigma^\star$, $M(w)$ halts within $c \cdot t(|w|)$ steps.

Def: A set $A \subseteq \Sigma^\star$ is in DSPACE$[s(n)]$ iff there exists a deterministic, multi-tape TM, M, and a constant c, such that,

1. $A \quad = \quad \mathcal{L}(M)$, and
2. $\forall w \in \Sigma^\star$, $M(w)$ uses at most $c \cdot s(|w|)$ work-tape cells.

(The input tape is considered "read-only" and not counted as space used.)

Thm: For any functions $t(n) \geq n,\ s(n) \geq \log n$,

$$\begin{aligned} \text{DTIME}[t(n)] &\subseteq \text{DSPACE}[t(n)] \\ \text{DSPACE}[s(n)] &\subseteq \text{DTIME}[2^{O(s(n))}] \end{aligned}$$

Proof: Let M be a DSPACE$[s(n)]$ TM, $w \in \Sigma_0^\star,\ n = |w|$

$M(w)$ has k tapes and uses at most $cs(n)$ work-tape cells.

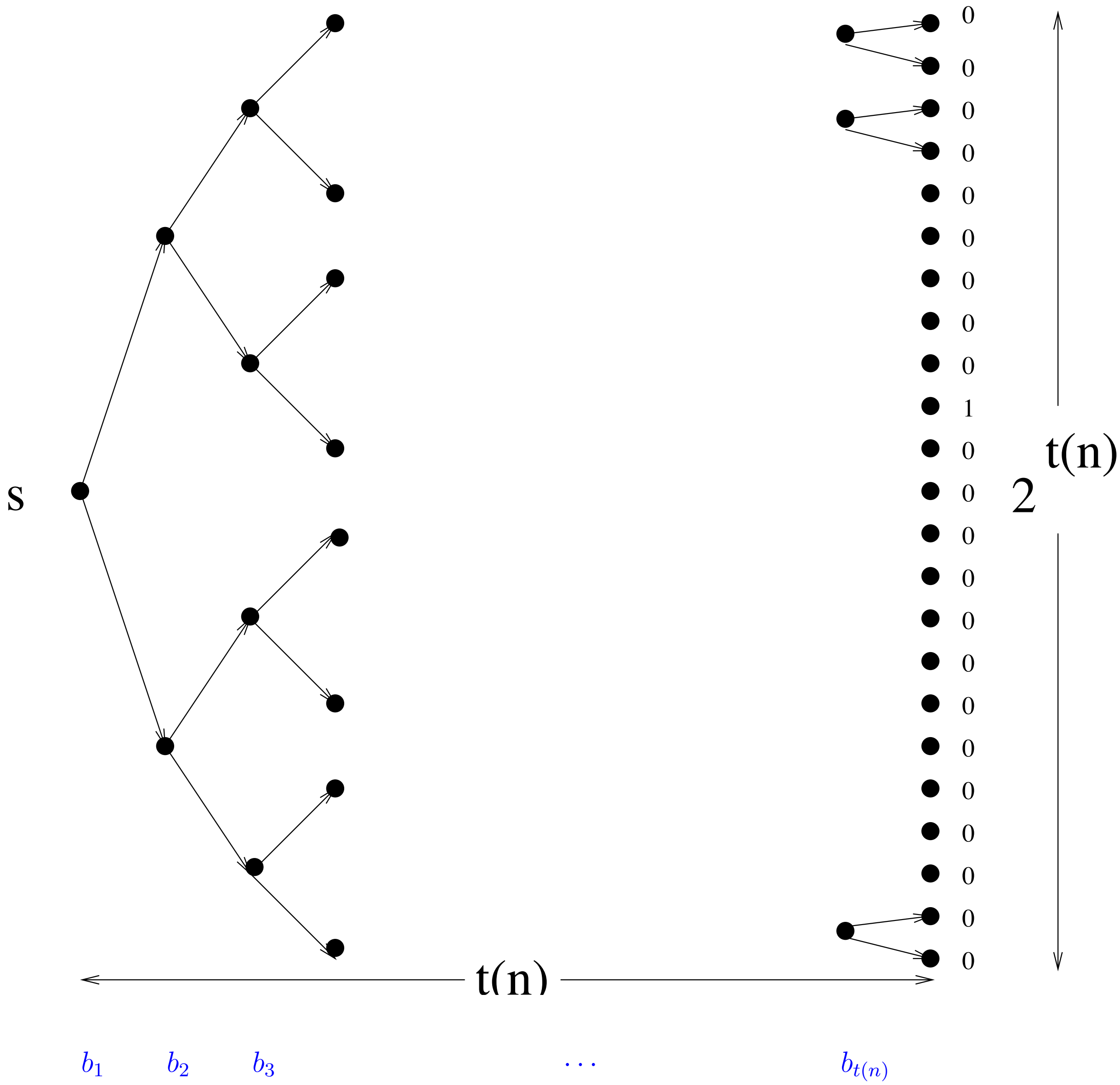
$M(w)$ has at most $2^{k's(n)}$ possible configurations:

$$\underbrace{|Q|}_{\text{\# states}} \cdot \underbrace{(n + cs(n) + 2)^k}_{\text{\# head positions}} \cdot \underbrace{|\Sigma|^{cs(n)}}_{\text{\# tape contents}} < 2^{k's(n)}$$

Thus, after $2^{k's(n)}$ steps, $M(w)$ must be in an infinite loop. $\square$

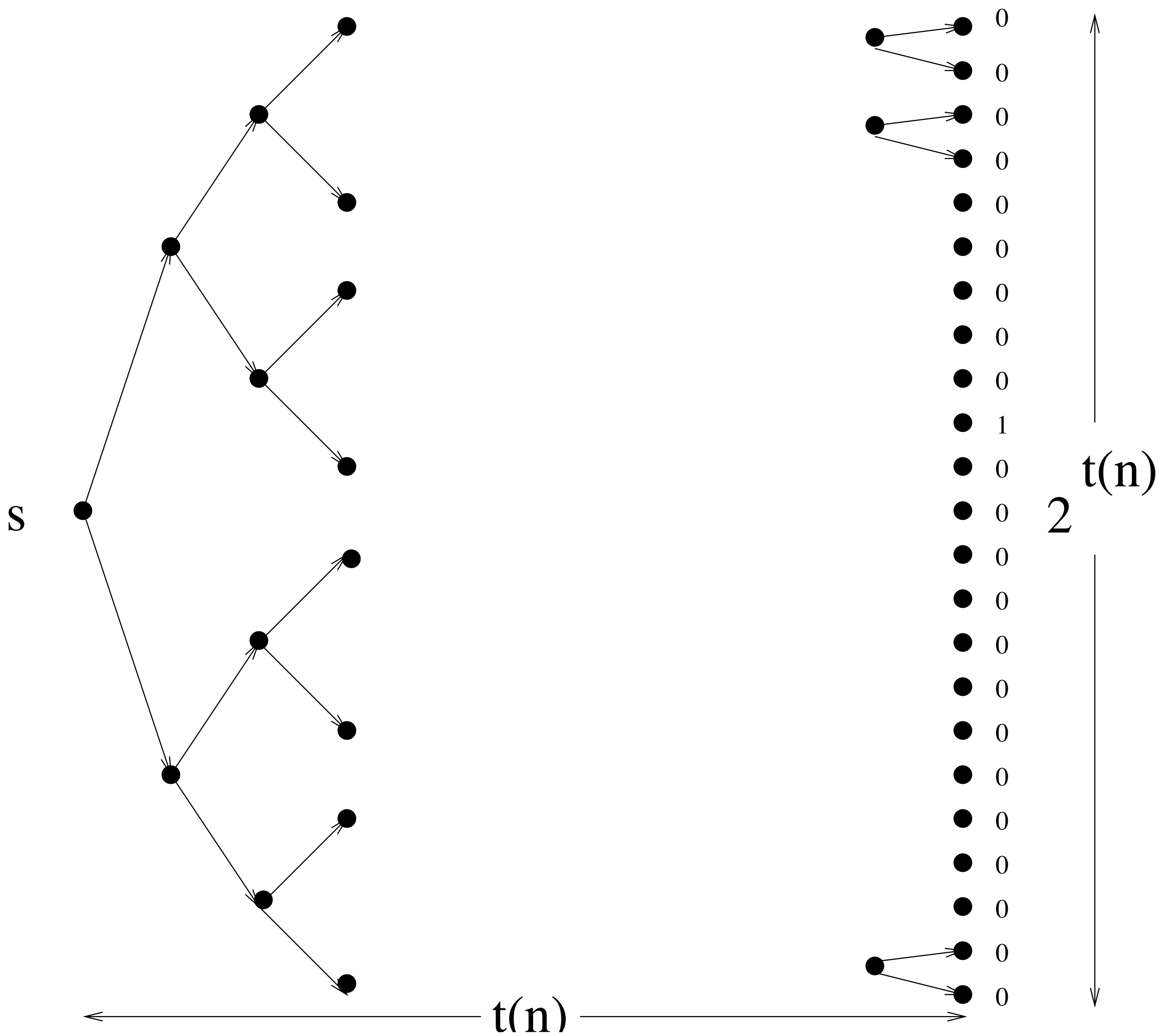
Theorem 8.1 *For any function $t(n) \geq n$,*

$$\text{DTIME}[t(n)] \subseteq \text{NTIME}[t(n)] \subseteq \text{DSPACE}[t(n)] \subseteq \text{DTIME}[2^{O(t(n))}]$$



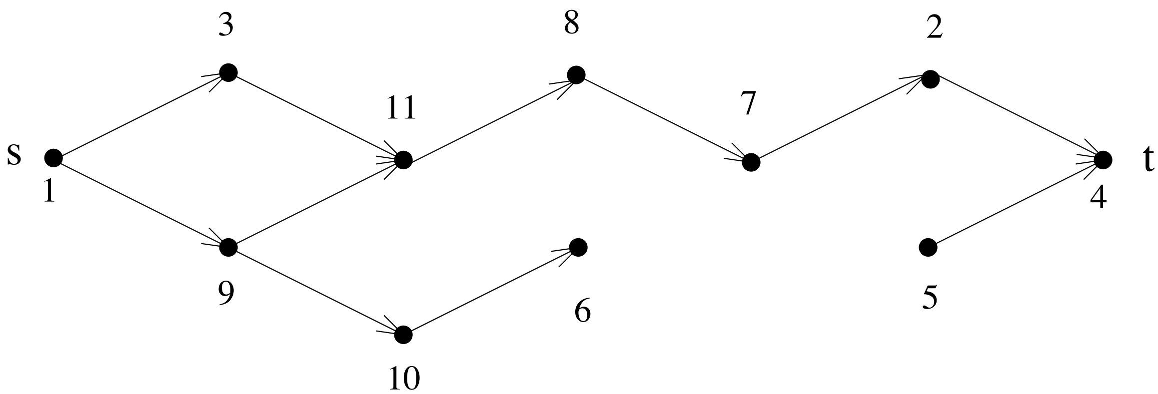
Corollary 8.2 L $\subseteq$ P $\subseteq$ NP $\subseteq$ PSPACE

NSPACE$[s(n)]$ is the set of problems accepted by NTMs using at most $O(s(n))$ space on each branch. [Can run in time $t(n) \leq 2^{O(s(n))}$.]



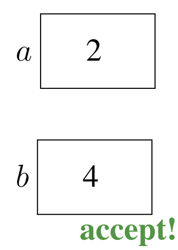
b_1 b_2 b_3 b_4 $\cdots$ $b_{t(n)}$

Definition 8.3 REACH $= \{G \mid s \xrightarrow{\star} t\}$ $\square$



Proposition 8.4 REACH $\in$ NL $=$ NSPACE$[\log n]$

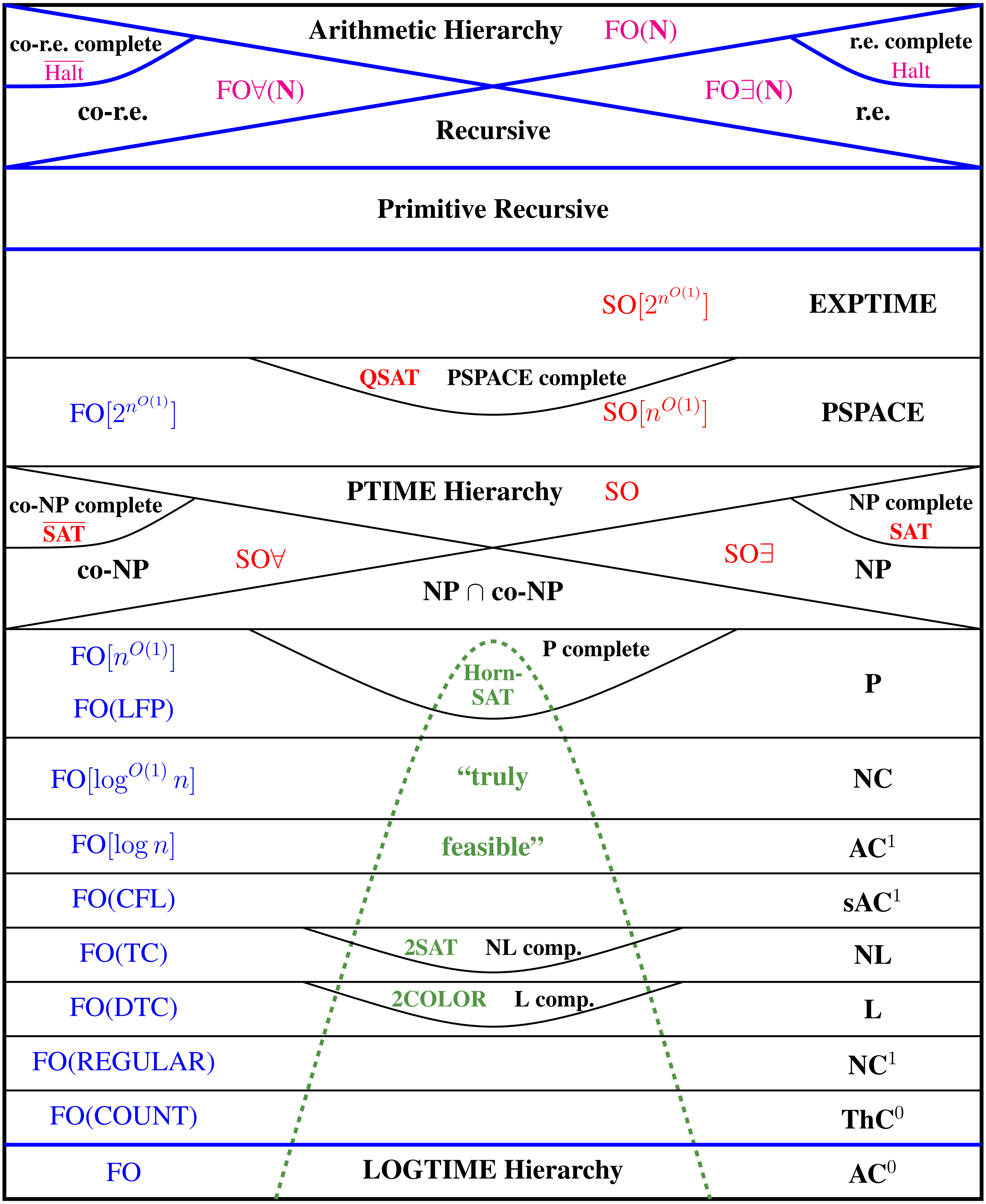
1. $b := s$
2. **for** $c := 1$ **to** $n = |V|$ **do** $\{$
3.   **if** $b = t$ **then accept**
4. $a := b$
5.   **choose** new b
6.   **if** $(\neg E(a, b))$ **then reject** $\}$
7. **reject**



Def: Problem T is **complete** for complexity class $\mathbf{C}$ iff

1. $T \in \mathbf{C}$, and
2. $\forall A \in \mathbf{C}\,(A \leq T)$

Reductions now must be in $F(\mathrm{L})$.


Arithmetic Hierarchy
FO(N)
co-r.e. complete
$\overline{\text{Halt}}$
r.e. complete
Halt
FO∀(N)
FO∃(N)
co-r.e.
r.e.
Recursive
Primitive Recursive
$\text{SO}[2^{n^{O(1)}}]$
EXPTIME
QSAT
PSPACE complete
$\text{FO}[2^{n^{O(1)}}]$
$\text{SO}[n^{O(1)}]$
PSPACE
PTIME Hierarchy
SO
co-NP complete
$\overline{\text{SAT}}$
NP complete
SAT
SO∀
SO∃
co-NP
NP
NP ∩ co-NP
$\text{FO}[n^{O(1)}]$
P complete
Horn-SAT
FO(LFP)
P
$\text{FO}[\log^{O(1)} n]$
“truly
NC
$\text{FO}[\log n]$
feasible”
AC^1
FO(CFL)
sAC^1
FO(TC)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
AC^0

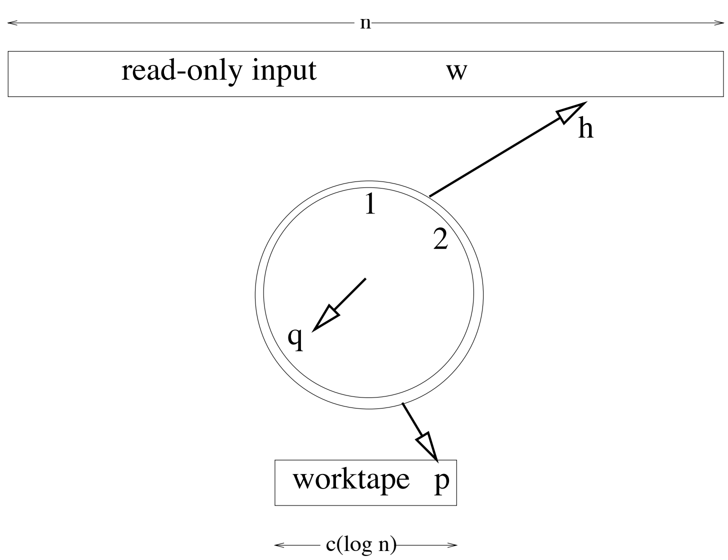
Thm: REACH is complete for NL.

Proof: Let $A \in \mathrm{NL}$, $A = \mathcal{L}(N)$, uses $c \log n$ bits of worktape.

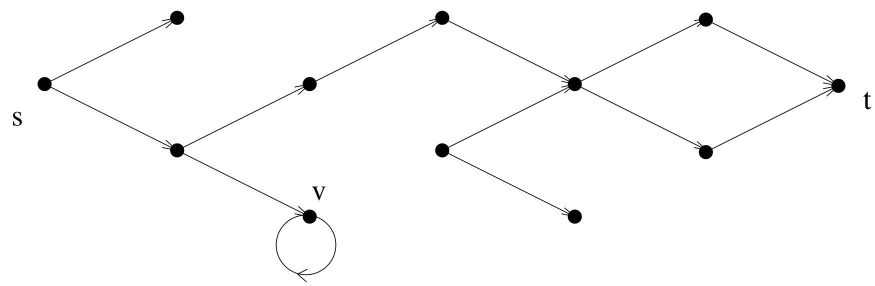
Input w, $\quad n = |w|$

$$w \;\mapsto\; \text{CompGraph}(N, w) \quad = \quad (V, E, s, t)$$

$$\begin{aligned}
V &= \left\{ \text{ID} = \langle q, h, p \rangle \;\middle|\; q \in \text{States}(N), h \leq n, |p| \leq c\lceil \log n \rceil \right\} \\
E &= \left\{ (\text{ID}_1, \text{ID}_2) \;\middle|\; \text{ID}_1(w) \underset{N}{\longrightarrow} \text{ID}_2(w) \right\} \\
s &= \text{initial ID} \\
t &= \text{accepting ID}
\end{aligned}$$



Claim: $w \in \mathcal{L}(N) \;\Leftrightarrow\;$ CompGraph$(N, w) \in$ REACH



$\square$

Cor: NL $\subseteq$ P

Proof: REACH $\in$ P

P is closed under (logspace) reductions.

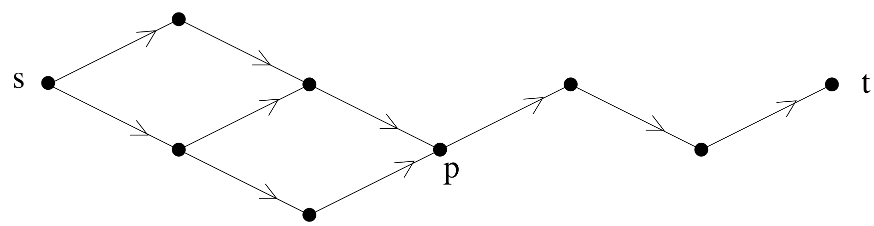
i.e., $(B \in \text{P} \quad \wedge \quad A \leq B) \quad \Rightarrow \quad A \in \text{P}$ $\square$

NSPACE vs. DSPACE

Proposition 8.5 NSPACE$[s(n)] \;\subseteq\;$ NTIME$[2^{O(s(n))}] \;\subseteq\;$ DSPACE$[2^{O(s(n))}]$

We can do much better!

Theorem 8.6 *[*Savich] $\quad \mathrm{REACH} \;\in\;$ DSPACE$[(\log n)^2]$



Proof:

$$\begin{aligned} G \in \mathrm{REACH} \quad &\Leftrightarrow \quad G \models \mathrm{PATH}(s,t,n) \\ \mathrm{PATH}(x,y,1) \quad &\equiv \quad x = y \;\vee\; E(x,y) \\ \mathrm{PATH}(x,y,2d) \quad &\equiv \quad \exists z\,(\mathrm{PATH}(x,z,d) \;\wedge\; \mathrm{PATH}(z,y,d)) \end{aligned}$$

$S_n(d)$ = space to check paths of dist. d in n-nodegraphs

$$\begin{aligned} S_n(n) \quad &= \quad \log n + S_n(n/2) \\ &= \quad O((\log n)^2) \end{aligned}$$

$\square$

Savitch's Thm: For $s(n) \geq \log n$,

$$\text{DSPACE}[s(n)] \;\subseteq\; \text{NSPACE}[s(n)] \;\subseteq\; \text{DSPACE}[(s(n))^2]$$

Proof: Let $A \in \text{NSPACE}[s(n)]$; $\quad A = \mathcal{L}(N)$

$$w \in A \qquad \Leftrightarrow \qquad \text{CompGraph}(N, w) \in \text{REACH}$$

$$|w| = n; \qquad |\text{CompGraph}(N, w)| = 2^{O(s(n))}$$

Testing if $\text{CompGraph}(N, w) \in \text{REACH}$ takes space,

$$\begin{aligned} (\log(|\text{CompGraph}(N, w)|))^2 &= (\log(2^{O(s(n))}))^2 \\ &= O((s(n))^2) \end{aligned}$$

From w build $\text{CompGraph}(N, w)$ in DSPACE[s(n)]. $\square$