# Parallel Computation and Inductive Definitions

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# Addition is First-Order

$$Q_{+} : \mathrm{STRUC}[\Sigma_{AB}] \to \mathrm{STRUC}[\Sigma_{s}]$$

$$\begin{array}{llcccccc}
A & & a_1 & a_2 & \ldots & a_{n-1} & a_n \\
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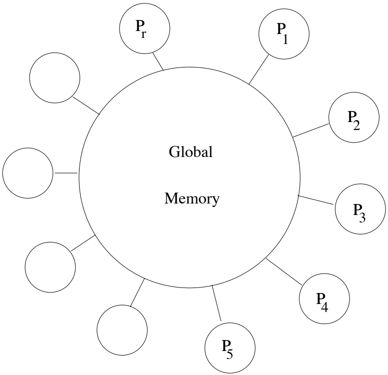
$$Q_+(i) \quad \equiv \quad A(i) \ \oplus \ B(i) \ \oplus \ C(i)$$

# Parallel Machines:

$\text{CRAM}[t(n)] \; = \text{CRCW-PRAM-TIME}[t(n)]\text{-HARD}[n^{O(1)}]$



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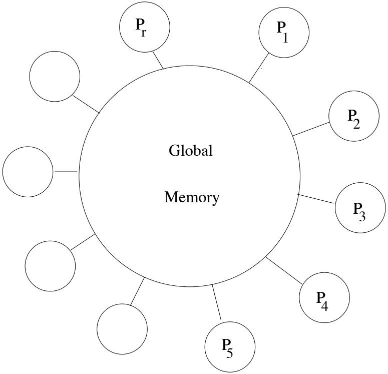

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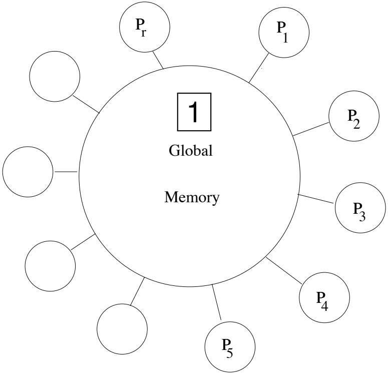

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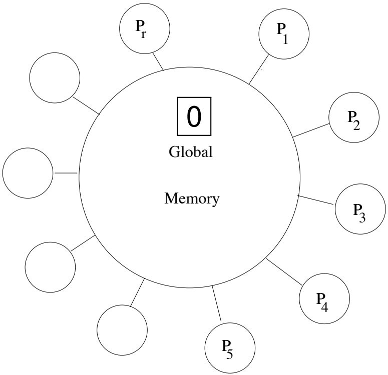


FO

=

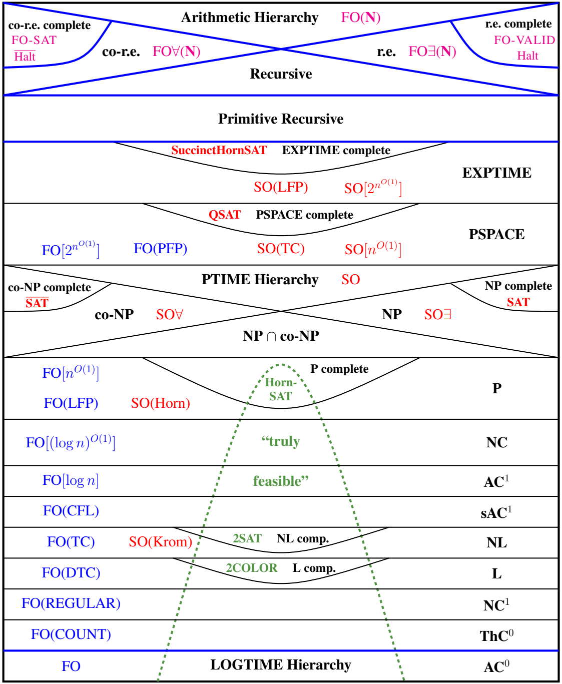
CRAM[1]

=

$AC^0$

=

Logarithmic-Time Hierarchy



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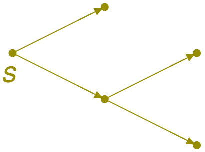


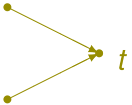

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$$E^{\star}(x,y) \stackrel{\text{def}}{=} x = y \lor E(x,y) \lor \exists z(E^{\star}(x,z) \land E^{\star}(z,y))$$

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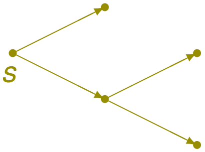


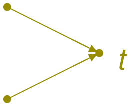

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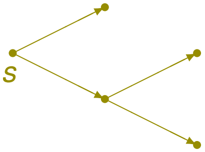


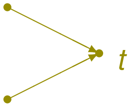

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$$G \in \text{REACH} \Leftrightarrow G \models (\mathrm{LFP}\varphi_{tc})(s,t) \qquad E^\star = (\mathrm{LFP}\varphi_{tc})$$

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Thus $I^t \subseteq F \quad$ and $\quad I^t = \mathrm{LFP}(\varphi)$. $\square$

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\mathrm{LFP}(\varphi_{tc}) & = & \varphi_{tc}^{\lceil 1+\log n\rceil}(\emptyset); \qquad \mathrm{REACH} \in \mathrm{IND}[\log n]
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(\varphi_{tc}^G)^{\lceil 1 + \log n \rceil}(\emptyset) & = & \big\{ (a, b) \in V^G \times V^G \ \big| \ \mathrm{dist}(a, b) \leq n \big\} \\
\mathrm{LFP}(\varphi_{tc}) & = & \varphi_{tc}^{\lceil 1 + \log n \rceil}(\emptyset); \qquad \mathrm{REACH} \in \mathrm{IND}[\log n]
\end{array}
$$

**Next we will show that** $\mathrm{IND}[t(n)] \ = \ \mathrm{FO}[t(n)]$.

# $\varphi_{tc}(R, x, y) \equiv x = y \vee E(x, y) \vee \exists z\,(R(x, z) \wedge R(z, y))$

1. Dummy universal quantification for base case:

$$\begin{aligned} \varphi_{tc}(R, x, y) &\equiv (\forall z.M_1)(\exists z)(R(x, z) \wedge R(z, y)) \\ M_1 &\equiv \neg(x = y \vee E(x, y)) \end{aligned}$$

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3. Requantify $x$ and $y$.

$$M_3 \equiv (x = u \wedge y = v)$$

$$\varphi_{tc}(R, x, y) \equiv [\,(\forall z.M_1)(\exists z)(\forall uv.M_2)(\exists xy.M_3)\,]\, R(x, y)$$

Every FO inductive definition is equivalent to a quantifier block.

## $\mathrm{QB}_{tc} \equiv [(\forall z.M_1)(\exists z)(\forall uv.M_2)(\forall xy.M_3)]$

$$\varphi_{tc}(R, x, y) \equiv [(\forall z.M_1)(\exists z)(\forall uv.M_2)(\exists xy.M_3)]R(x, y)$$

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Thus, for any structure $\mathcal{A} \in \mathrm{STRUC}[\Sigma_g]$,

$$\begin{aligned} \mathcal{A} \in \text{REACH} &\Leftrightarrow \mathcal{A} \models (\mathrm{LFP}\varphi_{tc})(s,t) \\ &\Leftrightarrow \mathcal{A} \models ([\mathrm{QB}_{tc}]^{\lceil 1+\log \|\mathcal{A}\| \rceil}\ \textbf{false})(s,t) \end{aligned}$$

$$
\begin{aligned}
\mathrm{CRAM}[t(n)] &= \text{concurrent parallel random access machine;}\\
&\quad\ \text{polynomial hardware, parallel time } O(t(n))\\[2ex]
\mathrm{IND}[t(n)] &= \text{first-order, depth } t(n) \text{ inductive definitions}\\[2ex]
\mathrm{FO}[t(n)] &= t(n) \text{ repetitions of a block of restricted quantifiers:}\\[2ex]
\mathrm{QB} &= [(Q_1 x_1 . M_1) \cdots (Q_k x_k . M_k)]; \quad M_i \text{ quantifier-free}\\[2ex]
\varphi_n &= \underbrace{[\mathrm{QB}][\mathrm{QB}] \cdots [\mathrm{QB}]}_{t(n)} M_0
\end{aligned}
$$

# parallel time = inductive depth = QB iteration

**Thm.** For all constructible, polynomially bounded $t(n)$,

$$\mathrm{CRAM}[t(n)] \;=\; \mathrm{IND}[t(n)] \;=\; \mathrm{FO}[t(n)]$$

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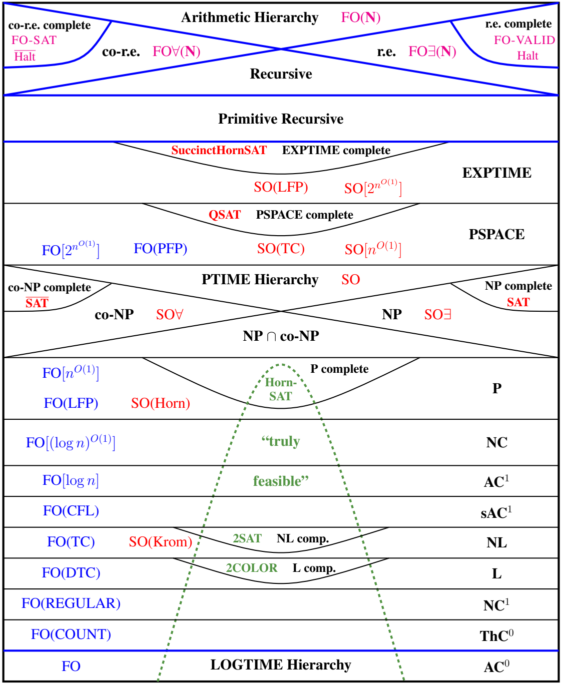
**Thm.** For all $t(n)$, even beyond polynomial,

$$\mathrm{CRAM}[t(n)] \quad = \quad \mathrm{FO}[t(n)]$$

For $t(n)$ poly bdd,

$$\mathrm{CRAM}[t(n)] = \mathrm{IND}[t(n)] = \mathrm{FO}[t(n)]$$



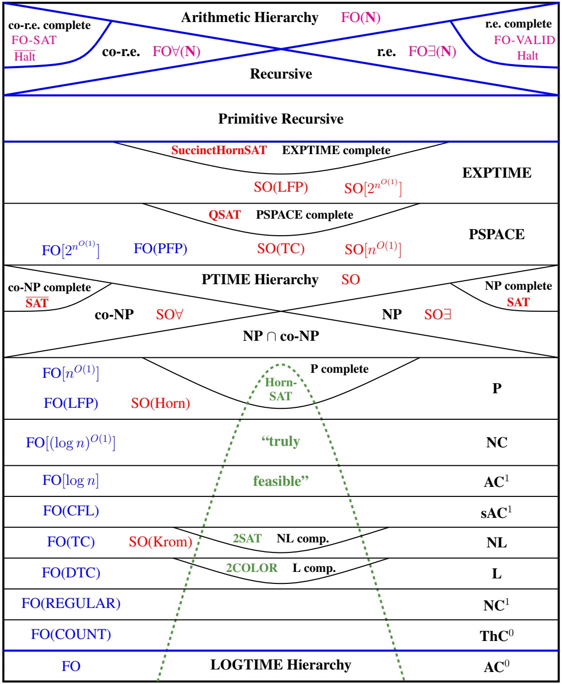
Remember that

for all $t(n)$,

$\mathrm{CRAM}[t(n)]$

$=$

$\mathrm{FO}[t(n)]$



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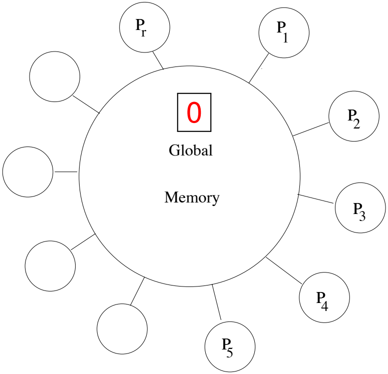
A second-order variable of arity $r$ is $n^r$ bits, corresponding to $2^{n^r}$ gates.

# SO: Parallel Machines with Exponential Hardware

Given $\varphi$ with $n$ variables and $m$ clauses, is $\varphi \in$ 3-SAT?



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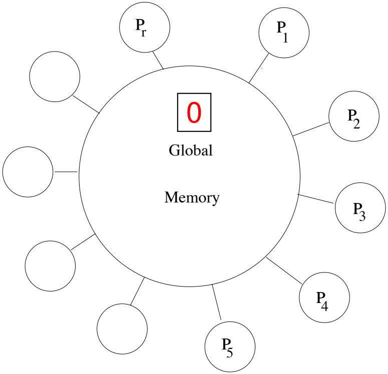

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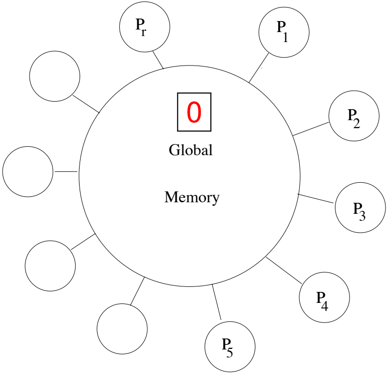

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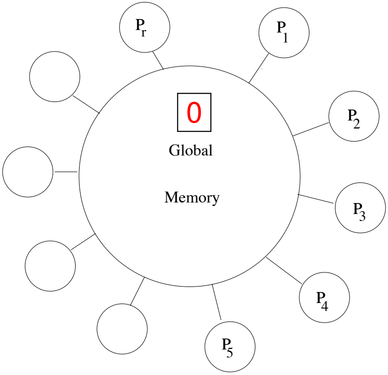

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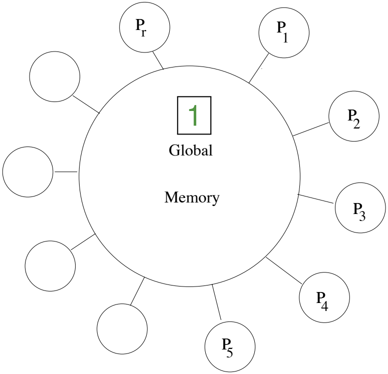

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# Parallel Time versus Amount of Hardware

$$\begin{aligned} \text{PSPACE} &= \text{FO}[2^{n^{O(1)}}] = \text{CRAM}[2^{n^{O(1)}}]\text{-HARD}[n^{O(1)}] \\ &= \text{SO}[n^{O(1)}] = \text{CRAM}[n^{O(1)}]\text{-HARD}[2^{n^{O(1)}}] \end{aligned}$$

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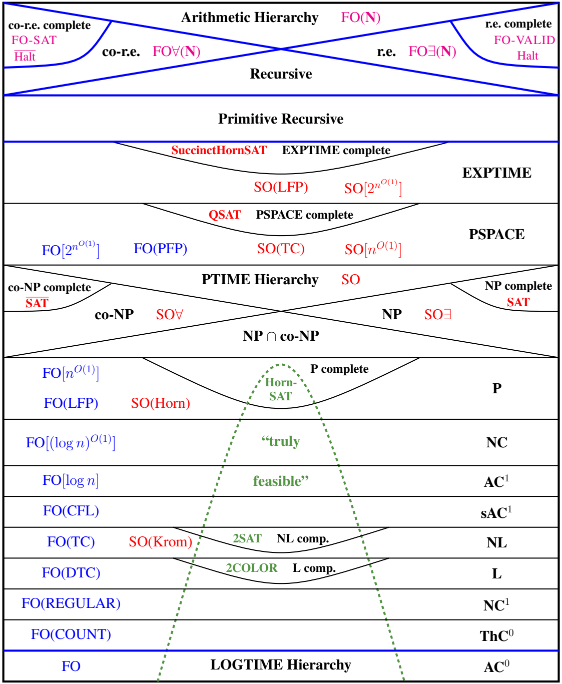
- We would love to understand this tradeoff.
- Is there such a thing as an inherently sequential problem?, i.e., is $\mathrm{NC} \neq \mathrm{P}$?
- Same tradeoff as number of variables vs. number of iterations of a quantifier block.

$$\mathrm{SO}[t(n)] = \mathrm{CRAM}[t(n)]\text{-}\mathrm{HARD}\text{-}[2^{n^{O(1)}}]$$



# Recent Breakthroughs in Descriptive Complexity

**Theorem** [Ben Rossman] Any first-order formula with any numeric relations ($\leq, +, \times, \ldots$) that means "I have a clique of size *k*" must have at least $k/4$ variables.

Creative new proof idea using Håstad's Switching Lemma gives the essentially optimal bound.

This lower bound is for a fixed formula, if it were for a sequence of polynomially-sized formulas, i.e., a fixed-point formula, it would follow that CLIQUE $\notin \mathrm{P}$ and thus $\mathrm{P} \neq \mathrm{NP}$.

Best previous bounds:

- *k* variables necessary and sufficient without ordering or other numeric relations [I 1980].
- Nothing was known with ordering except for the trivial fact that 2 variables are not enough.

# Recent Breakthroughs in Descriptive Complexity

**Theorem** [Martin Grohe] Fixed-Point Logic with Counting captures Polynomial Time on all classes of graphs with excluded minors.

Grohe proves that for every class of graphs with excluded minors, there is a constant $k$ such that two graphs of the class are isomorphic iff they agree on all $k$-variable formulas in fixed-point logic with counting.

Using Ehrenfeucht-Fraïssé games, this can be checked in polynomial time, $(O(n^k(\log n)))$. In the same time we can give a canonical description of the isomorphism type of any graph in the class. Thus every class of graphs with excluded minors admits the same general polynomial time canonization algorithm: we're isomorphic iff we agree on all formulas in $C_k$ and in particular, you are isomorphic to me iff your $C_k$ canonical description is equal to mine.

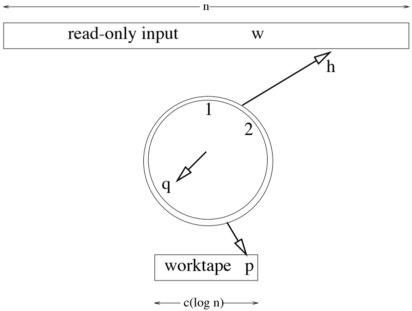
# Thm. REACH is complete for $\mathrm{NL} = \mathrm{NSPACE}[\log n]$.

**Proof:** Let $A \in \mathrm{NL}$, $A = \mathcal{L}(N)$, uses $c \log n$ bits of worktape.
Input $w$, $n = |w|$

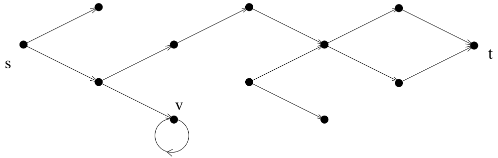
$$w \mapsto \mathrm{CompGraph}(N, w) \quad = \quad (V, E, s, t)$$

$$\begin{aligned}
V &= \left\{ \mathrm{ID} = \langle q, h, p \rangle \;\middle|\; q \in \mathrm{States}(N), h \leq n, |p| \leq c \lceil \log n \rceil \right\} \\
E &= \left\{ (\mathrm{ID}_1, \mathrm{ID}_2) \;\middle|\; \mathrm{ID}_1(w) \underset{N}{\longrightarrow} \mathrm{ID}_2(w) \right\} \\
s &= \text{initial ID} \\
t &= \text{accepting ID}
\end{aligned}$$

# NSPACE[log $n$] Turing Machine



**Claim.** $w \in \mathcal{L}(N) \iff \mathrm{CompGraph}(N, w) \in \mathsf{REACH}$




$\square$

**Cor:** $\mathrm{NL} \subseteq \mathrm{P}$

**Proof:** REACH $\in \mathrm{P}$

$\mathrm{P}$ is closed under (logspace) reductions.

i.e., $(B \in \mathrm{P} \quad \wedge \quad A \leq B) \quad \Rightarrow \quad A \in \mathrm{P}$ $\square$

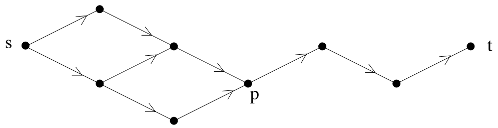
**Prop.**

$$\mathrm{NSPACE}[s(n)] \subseteq \mathrm{NTIME}[2^{O(s(n))}] \subseteq \mathrm{DSPACE}[2^{O(s(n))}]$$

**We can do much better!**

# Savitch's Theorem

$$\text{REACH} \in \text{DSPACE}(\log n)^2$$




**proof:**

$$\begin{array}{rcl} G \in \text{REACH} & \Leftrightarrow & G \models \text{PATH}_n(s,t) \\ \text{PATH}_1(x,y) & \equiv & x = y \ \vee \ E(x,y) \\ \text{PATH}_{2d}(x,y) & \equiv & \exists z \, (\text{PATH}_d(x,z) \ \wedge \ \text{PATH}_d(z,y)) \end{array}$$

$S_n(d)$ = space to check paths of dist. $d$ in $n$-nodegraphs

$$\begin{array}{rcl} S_n(n) & = & \log n + S_n(n/2) \\ & = & O((\log n)^2) \end{array}$$

$\square$

# Savitch's Theorem

$$\mathrm{DSPACE}[s(n)] \;\subseteq\; \mathrm{NSPACE}[n] \;\subseteq\; \mathrm{DSPACE}[(s(n))^2]$$

**proof:** Let $A \in \mathrm{NSPACE}[s(n)]$; $\quad A = \mathcal{L}(N)$

$$w \in A \qquad \Leftrightarrow \qquad \mathrm{CompGraph}(N, w) \in \text{REACH}$$

$$|w| = n; \qquad |\mathrm{CompGraph}(N, w)| = 2^{O(s(n))}$$

Testing if $\mathrm{CompGraph}(N, w) \in$ REACH takes space,

$$\begin{aligned} (\log(|\mathrm{CompGraph}(N, w)|))^2 &= (\log(2^{O(s(n))}))^2 \\ &= O((s(n))^2) \end{aligned}$$

From $w$ build $\mathrm{CompGraph}(N, w)$ in $\mathrm{DSPACE}$[s(n)]. $\square$

# $\overline{\text{REACH}} \in \mathrm{NL}$

**proof:** Fix $G$, let $N_d = \left|\{v \mid \text{distance}(s,v) \leq d\}\right|$

**Claim:** The following problems are in $\mathrm{NL}$:

1. $\mathrm{dist}(x,d)$: distance$(s,x) \leq d$
2. NDIST$(x,d;m)$: if $m = N_d$ then $\neg\mathrm{dist}(x,d)$

**proof:**

1. Guess the path of length $\leq d$ from $s$ to $x$.
2. Guess $m$ vertices, $v \neq x$, with $\mathrm{dist}(v,d)$.

$c := 0$;
**for** $v := 1$ to $n$ **do** $\{$ // nondeterministically
    ( $\mathrm{dist}(v,d)$ && $v \neq x$; $c{+}{+}$ ) $\quad ||$
    ( no-op )
$\}$
**if** $(c == m)$ **then** ACCEPT

$\square$

## **Claim.** We can compute $N_d$ in $\mathrm{NL}$.

**proof:** By induction on $d$.

**Base case:** $N_0 = 1$

**Inductive step:** Suppose we have $N_d$.

1. $c := 0;$
2. **for** $v := 1$ to $n$ **do** $\{$ // nondeterministically
3. $(\ \mathrm{dist}(v, d+1); c{+}{+})\quad ||$
4. $(\forall z\, (\mathrm{NDIST}(z, d; N_d)\ \vee\ (z \neq v \wedge \neg E(z, v))))$
5. $\}$
6. $N_{d+1} := c$

$\square$

$$G \in \overline{\mathrm{REACH}} \quad \Leftrightarrow \quad \mathrm{NDIST}(t, n; N_n) \qquad \square$$

**Thm.** $\mathrm{NSPACE}[s(n)] = \text{co-}\mathrm{NSPACE}[s(n)]$.

**proof:** Let $A \in \mathrm{NSPACE}[s(n)]$; $A = \mathcal{L}(N)$

$$w \in A \quad \Leftrightarrow \quad \mathrm{CompGraph}(N, w) \in \mathrm{REACH}$$

$$|w| = n; \qquad |\mathrm{CompGraph}(N, w)| = 2^{O(s(n))}$$

Testing if $\mathrm{CompGraph}(N, w) \in \overline{\mathrm{REACH}}$ takes space,

$$\begin{aligned} \log(|\mathrm{CompGraph}(N, w)|) &= \log(2^{O(s(n))}) \\ &= O(s(n)) \end{aligned} \qquad \square$$

# What We Know

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- **Major Missing Idea**: concept of work or conservation of energy in computation, i.e,

  in order to solve SAT or other hard problem we must do a certain amount of computational work.

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- $\mathrm{NC}^1 \subseteq \mathrm{FO}[\log n / \log\log n]$ and this is tight.
- Does REACH require $\mathrm{FO}[\log n]$? This would imply $\mathrm{NC}^1 \neq \mathrm{NL}$.

## Does It Matter? How important is P $\neq$ NP?

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- Basic trade-offs are not understood, e.g., trade-off between time and number of processors. Are any problems inherently sequential? How can we best use mulitcores?
- **SAT solvers** are impressive new general purpose problem solvers, e.g., used in model checking, AI planning, code synthesis. How good are current SAT solvers? How much can they be improved?

# Descriptive Complexity

**Fact:** For constructible $t(n)$, $\mathrm{FO}[t(n)] \;=\; \mathrm{CRAM}[t(n)]$

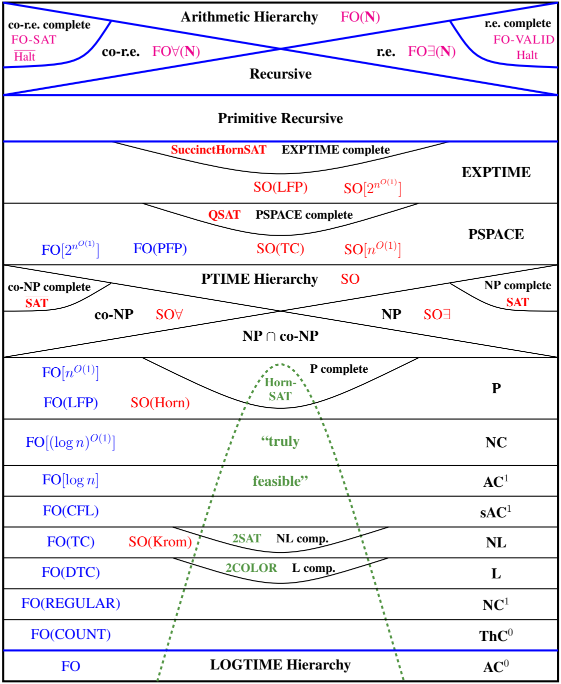
**Fact:** For $k = 1, 2, \ldots,$ $\mathrm{VAR}[k+1] \;=\; \mathrm{DSPACE}[n^k]$

The complexity of computing a query is closely tied to the complexity of describing the query.

$$\mathrm{P} = \mathrm{NP} \quad \Leftrightarrow \quad \mathrm{FO(LFP)} = \mathrm{SO}$$

$$\mathrm{ThC}^0 = \mathrm{NP} \quad \Leftrightarrow \quad \mathrm{FO(MAJ)} = \mathrm{SO}$$

$$\mathrm{P} = \mathrm{PSPACE} \quad \Leftrightarrow \quad \mathrm{FO(LFP)} = \mathrm{SO(TC)}$$


Arithmetic Hierarchy
FO(N)
co-r.e. complete
FO-SAT
$\overline{\text{Halt}}$
co-r.e.
FO∀(N)
r.e. complete
FO-VALID
Halt
r.e.
FO∃(N)
Recursive
Primitive Recursive
SuccinctHornSAT
EXPTIME complete
EXPTIME
SO(LFP)
$\text{SO}[2^{n^{O(1)}}]$
QSAT
PSPACE complete
PSPACE
$\text{FO}[2^{n^{O(1)}}]$
FO(PFP)
SO(TC)
$\text{SO}[n^{O(1)}]$
PTIME Hierarchy
SO
co-NP complete
$\overline{\text{SAT}}$
co-NP
SO∀
NP complete
SAT
NP
SO∃
NP ∩ co-NP
$\text{FO}[n^{O(1)}]$
P complete
Horn-SAT
P
FO(LFP)
SO(Horn)
$\text{FO}[(\log n)^{O(1)}]$
“truly
NC
$\text{FO}[\log n]$
feasible”
$\text{AC}^1$
FO(CFL)
$\text{sAC}^1$
FO(TC)
SO(Krom)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
$\text{NC}^1$
FO(COUNT)
$\text{ThC}^0$
FO
LOGTIME Hierarchy
$\text{AC}^0$