Fundamental Th. of Reductions: If $S \leq T$, then,

1. If T is r.e., then S is r.e..
2. If T is co-r.e., then S is co-r.e..
3. If T is Recursive, then S is Recursive.

Proof: $S \leq T \ \wedge \ T \in$ r.e. $\quad \Rightarrow \quad S \in$ r.e.

Let $f : S \leq T$, i.e., $\forall x (x \in S \ \Leftrightarrow \ f(x) \in T)$, $\qquad T = W_i$.

From M_i compute the TM $M_{i'}$ which on input x does the following: (a). compute $f(x)$; (b) run $M_i(f(x))$

$$\boxed{\quad M_{i'} \quad} = \boxed{f}\boxed{\quad M_i \quad}$$

$$(x \in S) \ \Leftrightarrow \ (f(x) \in T) \ \Leftrightarrow \ (M_i(f(x)) = 1) \ \Leftrightarrow \ (M_{i'}(x) = 1)$$

Therefore, $S = W_{i'}$, and S is r.e. as desired.

In other words, $\mathrm{P}_S = p_T \circ f$. We are given the Turing machines that compute the partial recursive function p_T and the total recursive function f. From these, we can easily construct the Turing machine, $M_{i'}$, which computes p_S.

Observation 4.1 $f : S \leq T \quad \Leftrightarrow \quad f : \overline{S} \leq \overline{T}$.

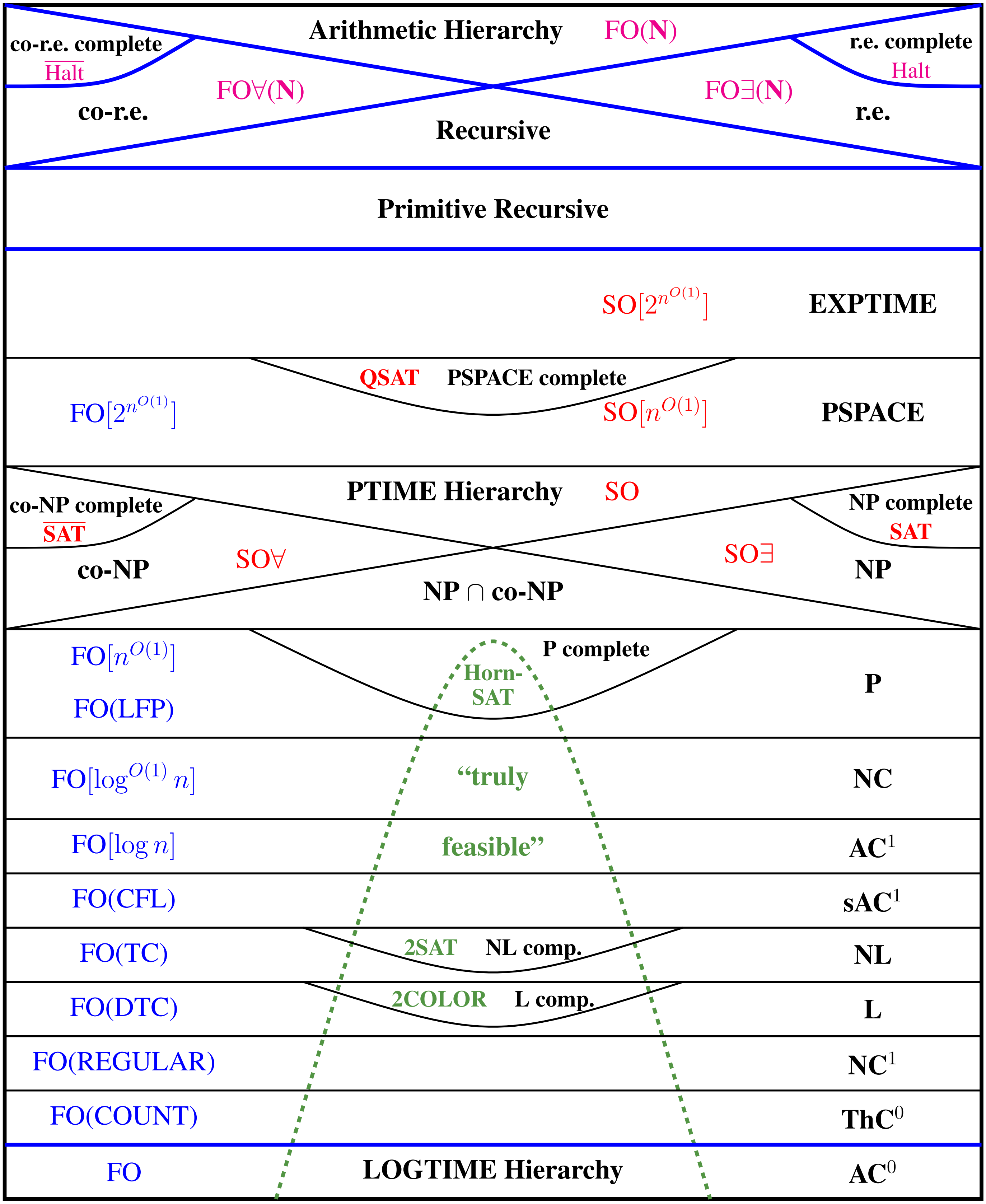
Thus, $\qquad T \in$ co-r.e. $\Rightarrow \overline{T} \in$ r.e. $\Rightarrow \overline{S} \in$ r.e. $\Rightarrow S \in$ co-r.e.

$T \in$ Recursive $\quad \Rightarrow \quad (T \in$ r.e. $\wedge \ T \in$ co-r.e.$) \quad \Rightarrow$

$(S \in$ r.e. $\wedge \ S \in$ co-r.e.$) \quad \Rightarrow \quad S \in$ Recursive $\qquad \square$

Moral: Suppose $S \leq T$. Then,

- If T is easy, then so is S.
- If S is hard, then so is T.


Arithmetic Hierarchy
FO(N)
co-r.e. complete
$\overline{\text{Halt}}$
r.e. complete
Halt
FO∀(N)
FO∃(N)
co-r.e.
r.e.
Recursive
Primitive Recursive
$SO[2^{n^{O(1)}}]$
EXPTIME
QSAT
PSPACE complete
$FO[2^{n^{O(1)}}]$
$SO[n^{O(1)}]$
PSPACE
PTIME Hierarchy
SO
co-NP complete
$\overline{\text{SAT}}$
NP complete
SAT
SO∀
SO∃
co-NP
NP
NP ∩ co-NP
$FO[n^{O(1)}]$
P complete
Horn-SAT
P
FO(LFP)
$FO[\log^{O(1)} n]$
"truly
NC
$FO[\log n]$
feasible"
AC^1
FO(CFL)
sAC^1
FO(TC)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
AC^0

Proposition 4.2 *For $S \subseteq \mathbf{N}$ or $S \subseteq \{0,1\}^\star$, S is r.e. complete iff $\overline{S}$ is co-r.e. complete.*

Proof: Suppose S is r.e. complete. Thus, $S \in$ r.e. and $\forall B \in$ r.e. $B \leq S$.

Thus $\overline{S} \in$ co-r.e.. Also, for all $B \in$ r.e., $B \leq S$. Thus, $\overline{B} \leq \overline{S}$.

Thus, all co-r.e. sets are reducible to $\overline{S}$. Thus $\overline{S}$ is r.e. complete.

The proof of the converse is similar, i.e., essentially identical. $\square$

The **Arithmetic Hierarchy** is at the top of the World-of-Computability-and-Complexity diagram.

Definition 4.3 Let $S \subseteq \mathbf{N}$. S is an element of Σ_k iff there is a decidable predicate φ, such that,

$$S = \{n \mid (\exists x_1)(\forall x_2)\cdots(Q_k x_k)\varphi(n, x_1, \ldots, x_k)\},$$

here Q_k is $\forall$ if k is even and $\exists$ if k is odd.

Similarly, S is an element of Π_k iff,

$$S = \{n \mid (\forall x_1)(\exists x_2)\cdots(Q'_k x_k)\psi(n, x_1, \ldots, x_k)\},$$

for some decidable predicate ψ. Here Q'_k is $\forall$ if k is odd and $\exists$ if k is even.

Define the Arithmetic Hierarchy (AH) to be $\bigcup_{k=1}^{\infty} \Sigma_k$. Note that AH is thus also equal to $\bigcup_{k=1}^{\infty} \Pi_k$. $\square$

Proposition 4.4 $\Sigma_1 =$ r.e. *and* $\Pi_1 =$ *co*-r.e..

Proof: Let W_i be an arbitrary r.e. set. Observe that

$$W_i = \{n \mid \exists c \in \mathbf{N} \quad \text{COMP}(i, n, c, 1)\} .$$

Here COMP(i, n, c, y) is the very useful decidable predicate meaning that c is an encoding of a complete halting computation of TM $M_i(n)$ and the output is y, i.e., $M_i(n) = y$.

Thus, $W_i \in \Sigma_1$.

Conversely, suppose that $S \in \Sigma_1$, i.e.,

$S = \{n \mid (\exists x)\, \varphi(n, x)\}$, for a decidable predicate, φ.

We can build a TM, M_S which accepts exactly S by doing the following:

for $x = 0$ **to** ∞, **if** $(\varphi(n, x))$: **return**(1).

Thus, we have shown that r.e. $= \Sigma_1$.

From the definition of Σ_1 and Π_1 it follows that $\Sigma_1 =$ co-Π_1. Thus, co-r.e. $= \Pi_1$. $\square$