Lecture 14: PSPACE

$$\text{PSPACE} \quad = \quad \text{DSPACE}[n^{O(1)}] \quad = \quad \text{NSPACE}[n^{O(1)}] \quad = \quad \text{ATIME}[n^{O(1)}]$$

- PSPACE consists of what we could compute with a feasible amount of hardware, but with no time limit.
- PSPACE is a large and very robust complexity class.
- With polynomially many bits of memory, we can search any implicitly-defined graph of exponential size. This leads to complete problems such as reachability on exponentially-large graphs.
- We can search the game tree of any board game whose configurations are describable with polynomially-many bits and which lasts at most polynomially many moves. This leads to complete problems concerning winning strategies.

PSPACE-Complete Problems

Recall PSPACE $=$ ATIME$[n^{O(1)}]$

Recall QSAT, the quantified satisfiability problem.

Proposition 14.1 QSAT *is* PSPACE*-complete.*

Proof: We've already seen that QSAT $\in$ ATIME$[n] \subseteq$ PSPACE.

QSAT is hard for ATIME$[n^k]$:

Let M be an ATIME$[n^k]$ TM, w an input, $n = |w|$

Let M write down its n^k alternating choices, $c_1\ c_2 \ldots c_{n^k}$.

Deterministic TM D evaluates the answer, i.e., for all inputs w, $M(w) = 1 \;\Leftrightarrow\; \exists c_1 \forall c_2 \cdots \exists c_{n^k} \, (D(\bar{c}, w) = 1)$

By Cook's Theorem $\exists$ reduction $f : \mathcal{L}(D) \leq$ SAT:

$$D(\bar{c}, w) = 1 \quad \Leftrightarrow \quad f(\bar{c}, w) \in \text{SAT}$$

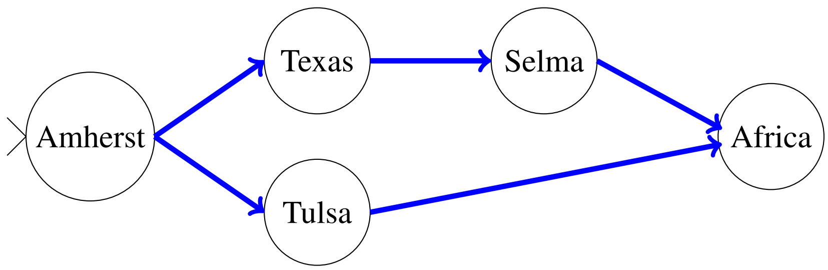
Let the new boolean variables in $f(\bar{c}, w)$ be $d_1 \ldots d_{t(n)}$.

$$M(w) = 1 \quad \Leftrightarrow \quad \text{“}\exists c_1 \forall c_2 \cdots \exists c_{n^k} \, d_1 \ldots d_{t(n)} \, (f(\bar{c}, w))\text{”} \in \text{QSAT} \qquad \square$$

Geography is a two-person game.

1. E "chooses" the start vertex v_1.
2. A chooses v_2, having an edge from v_1
3. E chooses v_3, having an edge from v_2, etc.

No vertex may be chosen twice. Whoever moves last wins.



Let GEOGRAPHY be the set of positions in geography games s.t. $\exists$ has a winning strategy.

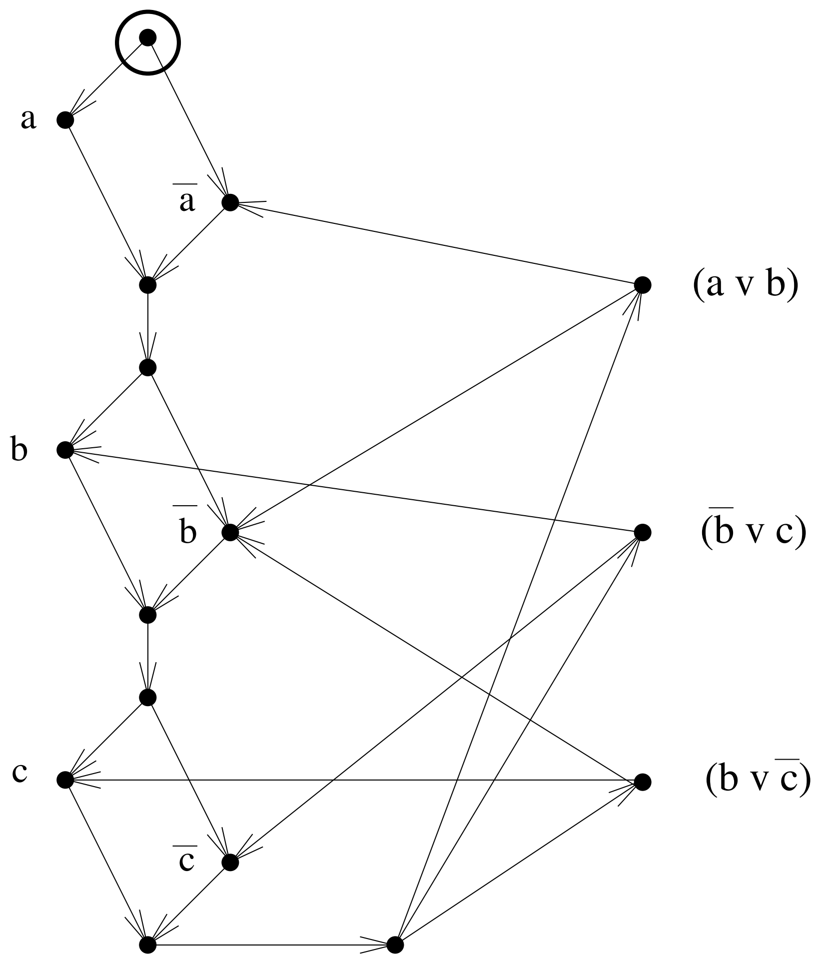
Proposition 14.2 GEOGRAPHY *is* PSPACE-*complete.*

Proof: GEOGRAPHY $\in$ PSPACE: search the polynomial-depth game tree. A polynomial-size stack suffices.

Show: QSAT $\leq$ GEOGRAPHY

Given formula, φ, build graph G_φ s.t. $\exists$ chooses existential variables; $\forall$ chooses universal variables.

$$\varphi \equiv \exists a \, \forall b \, \exists c \; [(a \vee b) \; \wedge \; (\bar{b} \vee c) \; \wedge \; (b \vee \bar{c})]$$



$\square$

Definition 14.3 A **succinct** representation of a graph is $G(n, C, s, t) = (V, E, s, t)$ where C is a boolean circuit with $2n$ inputs and

$$V = \{w \mid w \in \{0,1\}^n\}$$
$$E = \{(w, w') \mid C(w, w') = 1\}$$

□

$$\text{SUCCINCT REACH} = \{(n, C, s, t) \mid G(n, C, s, t) \in \text{REACH}\}$$

Proposition 14.4 SUCCINCT REACH $\in$ PSPACE

Why?

Remember Savitch's Thm:

$$\text{REACH} \in \text{NSPACE}[\log n] \subseteq \text{DSPACE}[(\log n)^2]$$

$$\text{SUCCINCT REACH} \in \text{NSPACE}[n] \subseteq \text{DSPACE}[n^2] \subseteq \text{PSPACE}$$

□

Proposition 14.5 SUCCINCT REACH *is* PSPACE*-complete.*

Proof: Let M be a DSPACE$[n^k]$ TM, input w, $n = |w|$

$$M(w) = 1 \quad \leftrightarrow \quad \text{CompGraph}(M, w) \in \text{REACH}$$

$$\text{CompGraph}(n, w) = (V, E, s, t)$$

$$V = \{\text{ID} = \langle q, h, p\rangle \mid q \in \text{States}(N), h \leq n, |p| \leq c\, n^k\}$$
$$E = \{(\text{ID}_1, \text{ID}_2) \mid \text{ID}_1(w) \underset{M}{\longrightarrow} \text{ID}_2(w)\}$$
$$s = \text{initial ID}$$
$$t = \text{accepting ID}$$

□

Succinct Representation of CompGraph(n, w):

$$
\begin{aligned}
V &= \left\{\text{ID} = \langle q, h, p\rangle \;\middle|\; q \in \text{States}(N), h \leq n, |p| \leq c\,n^k\right\} \\
E &= \left\{(\text{ID}_1, \text{ID}_2) \;\middle|\; \text{ID}_1(w) \underset{M}{\longrightarrow} \text{ID}_2(w)\right\}
\end{aligned}
$$

Let $V = \{0,1\}^{c'n^k}$

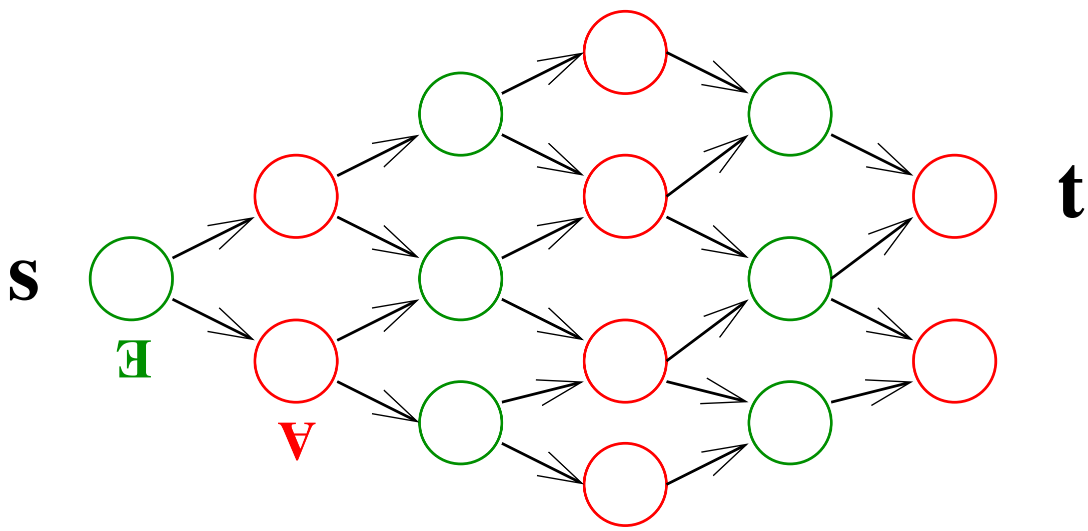
Build circuit C_w: on input $u, v \in V$, accept iff $u \underset{M}{\longrightarrow} v$.

$$
M(w) = 1 \quad \Leftrightarrow \quad G(c'n^k, C_w, s, t) \in \text{SUCCINCT REACH} \qquad \square
$$

The vertices of an **alternating graph**, $G = (V, A, E)$, are split into: **existential vertices** and **universal vertices**.

Def: vertex t is **reachable** from vertex s in G iff

1. $s = t$, or
2. s is existential and for some edge, $\langle s, a \rangle \in E$, t is reachable from a, or,
3. s is universal and there is an edge leaving s and for all edges, $\langle s, a \rangle \in E$, t is reachable from a.



Def: Let

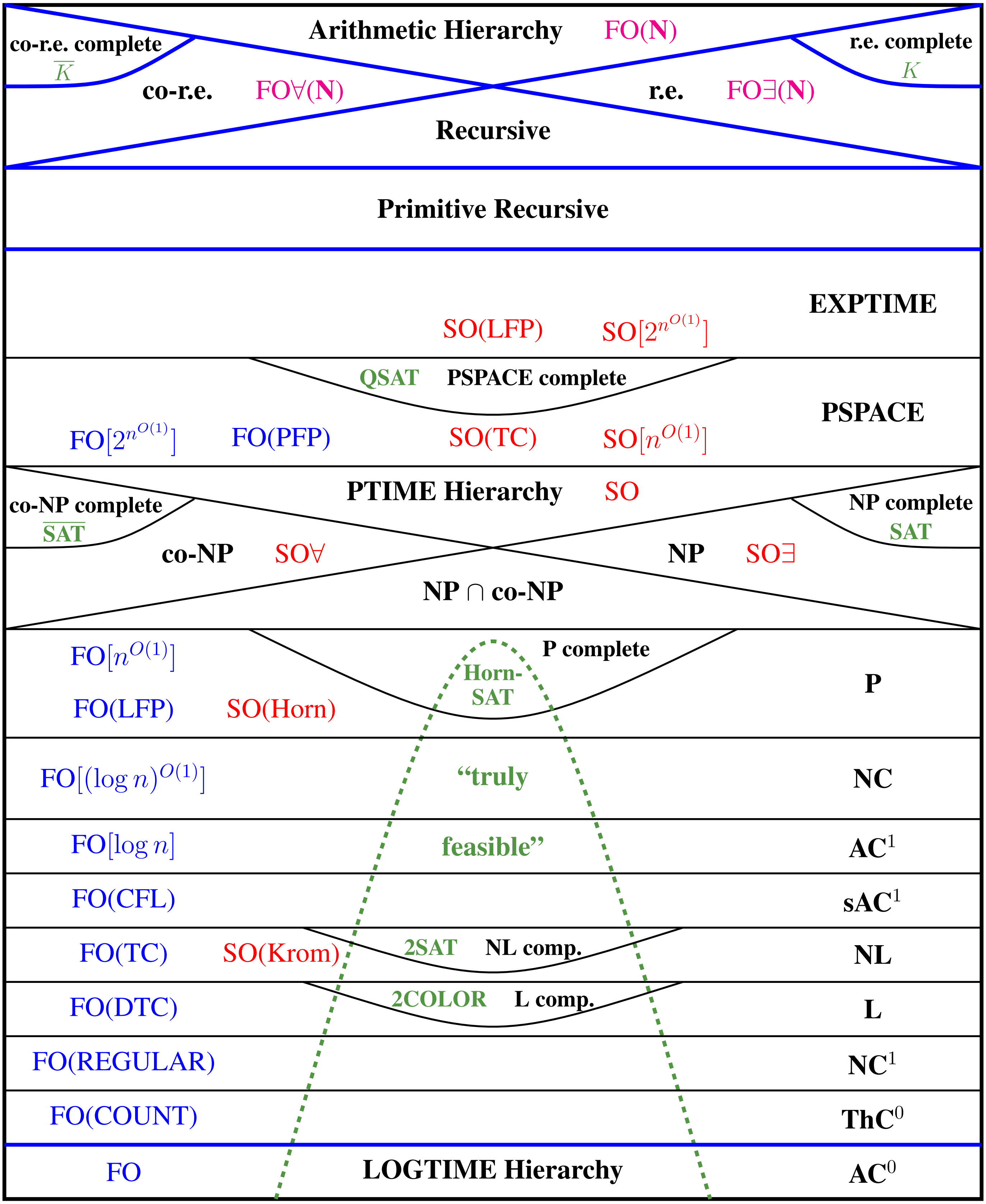
$$\text{AREACH} = \{G = (V, A, E, s, t) \mid t \text{ is reachable from } s\}$$

Prop: AREACH is P complete.

Proof: Think about it! (Very similar to the proof that REACH is NL complete.) □

Cor: CVP and MCVP are P complete.

Proof: Easy to see that AREACH $\leq$ MCVP $\leq$ CVP. □


Arithmetic Hierarchy
FO(N)
co-r.e. complete
$\overline{K}$
r.e. complete
K
co-r.e.
FO∀(N)
r.e.
FO∃(N)
Recursive
Primitive Recursive
EXPTIME
SO(LFP)
$SO[2^{n^{O(1)}}]$
QSAT
PSPACE complete
PSPACE
$FO[2^{n^{O(1)}}]$
FO(PFP)
SO(TC)
$SO[n^{O(1)}]$
PTIME Hierarchy
SO
co-NP complete
$\overline{SAT}$
NP complete
SAT
co-NP
SO∀
NP
SO∃
NP ∩ co-NP
$FO[n^{O(1)}]$
P complete
Horn-SAT
P
FO(LFP)
SO(Horn)
$FO[(\log n)^{O(1)}]$
“truly
NC
$FO[\log n]$
feasible”
AC^1
FO(CFL)
sAC^1
FO(TC)
SO(Krom)
2SAT
NL comp.
NL
FO(DTC)
2COLOR
L comp.
L
FO(REGULAR)
NC^1
FO(COUNT)
ThC^0
FO
LOGTIME Hierarchy
AC^0