CS601:	**Homework 2**	Due in class Thursday, Feb. 14, 2019

Please do all the assigned reading and read all the lecture notes posted on the syllabus page.

Problems

1. Recall that HALT $= \{(i, j) \mid M_i(j) \downarrow\}$. Show that HALT is r.e. complete by building reductions, $f_1 : \text{HALT} \leq A_{0,17}$, and $f_2 : A_{0,17} \leq \text{HALT}$. You should argue simply and clearly why your f_i's are total easy-to-compute functions and why they satisfy the defining conditions of reductions, e.g.,

$$\forall x \in \mathbf{N} \quad (x \in A_{0.17}) \quad \Leftrightarrow \quad (f_2(x) \in \text{HALT})\ .$$

   [Note that f_1 is a map from inputs to HALT to inputs to $A_{0,17}$ and f_2 is a map from inputs to $A_{0,17}$ to inputs to HALT. Thus, f_1 is given a pair (i, j) and outputs a natural; whereas f_2 is given a natural and outputs a pair of naturals.]

2. In a Tiling problem, we are given an integer in unary 1^n, a set of square tile types, $T = \{t_0, \ldots, t_k\}$, together with two relations $H, V \subseteq T \times T$, where $H(t_1, t_2)$ means that t_2 can be placed immediately to the right of t_1 and $V(t_1, t_2)$ means that t_2 can be placed immediately below t_1. TILING is the problem of deciding whether there exists an $n \times n$ grid of tiles satisfying the horizontal and vertical constraints and with t_0 in the upper left position.

   Show that TILING is NP complete.

   [Hint: you can encode a Turing machine computation via such tiles. Let N be a nondeterministic TM. Consider a set of tiles that are labeled with the tape symbols from Σ plus an additional set of tiles that are labeled with triples $\langle q, a, \delta\rangle$ where q is a state, $a \in \Sigma$ and $\delta \in \{0, 1\}$ indicates which nondeterministic choice the machine will make at this step, plus some additional tiles.

$\triangleright$	q_0, a, δ_0	$b; q$	c	$\sqcup$
$\triangleright$	d	q, b, δ	$c; q'$	$\sqcup$
$\triangleright$	d	e	q', c, δ'	$\sqcup$

   Suppose for example that choice δ of N in state q looking at a b is to write e, and go to the right into state q'. Then the tile marked $\langle q, b, \delta\rangle$ would have a signal at its bottom indicating that the next tile should contain symbol e and it could have a signal on its right indicating that the cell on its right should have a signal on its bottom indicating that the cell below should be in state q'.]

3. Define the following series of sets in the Arithmetic Hierarchy:

$$\begin{aligned}
T_1 &= \{n \mid \exists c\, \text{COMP}(n, n, c, 1)\} \\
T_2 &= \{n \mid \exists x_1 \forall c \neg\, \text{COMP}(n, \langle n, x_1\rangle, c, 1)\} \\
T_3 &= \{n \mid \exists x_1 \forall x_2 \exists c\, \text{COMP}(n, \langle n, x_1, x_2\rangle, c, 1)\} \\
T_{2k+1} &= \{n \mid \exists x_1 \forall x_2 \exists x_3 \cdots \forall x_{2k} \exists c\, \text{COMP}(n, \langle n, x_1, x_2, \ldots, x_{2k}\rangle, c, 1)\} \\
T_{2k+2} &= \{n \mid \exists x_1 \forall x_2 \exists x_3 \cdots \exists x_{2k+1} \forall c \neg\, \text{COMP}(n, \langle n, x_1, x_2, \ldots, x_{2k+1}\rangle, c, 1)\}
\end{aligned}$$

[Here we are using notation for k-tuples: $\langle x, y, z, \cdots \rangle \stackrel{\text{def}}{=} \langle x, \langle y, z, \cdots \rangle \rangle$, where $\langle x, y \rangle \stackrel{\text{def}}{=} P(x, y)$. The details of the encoding are not important; just that it is computable to encode and decode tuples.

COMP(n, w, c, y) means that $M_n(w) = y$ and c is a complete encoding of this computation.]

Prove that for all k, T_k is complete for Σ_k.

[Hint: don't panic. Prove this first for T_1. Then prove it for T_2. If you want to quit there, you will get two thirds credit for this problem. The other third involves just keeping your cool and dealing with the notation.

Let

$$S \quad = \quad \left\{ n \;\middle|\; (\exists x_1)(\forall x_2) \cdots (Q_k x_k) \varphi(n, x_1, \ldots, x_k) \right\},$$

be an arbitrary Σ_k set, where φ is a computable predicate.

Your job is to show that $S_k \leq T_k$.]