Please do all the assigned reading and read all the lecture notes posted on the syllabus page.

Problems

1. Recall that HALT = \{(i, j) \mid M_i(j) \downarrow\}. Show that HALT is r.e. complete by building reductions, 
\( f_1 : \text{HALT} \leq A_{0,17} \), and \( f_2 : A_{0,17} \leq \text{HALT} \). You should argue simply and clearly why your \( f_i \)'s are total easy-to-compute functions and why they satisfy the defining conditions of reductions, e.g.,

\[ \forall x \in \mathbb{N} \ (x \in A_{0,17}) \iff (f_2(x) \in \text{HALT}) \ . \]

[Note that \( f_1 \) is a map from inputs to HALT to inputs to \( A_{0,17} \) and \( f_2 \) is a map from inputs to \( A_{0,17} \) to inputs to HALT. Thus, \( f_1 \) is given a pair \((i, j)\) and outputs a natural; whereas \( f_2 \) is given a natural and outputs a pair of naturals.]

2. In a Tiling problem, we are given an integer in unary \( n \), a set of square tile types, \( T = \{t_0, \ldots, t_k\} \), together with two relations \( H, V \subseteq T \times T \), where \( H(t_1, t_2) \) means that \( t_2 \) can be placed immediately to the right of \( t_1 \) and \( V(t_1, t_2) \) means that \( t_2 \) can be placed immediately below \( t_1 \). TILING is the problem of deciding whether there exists an \( n \times n \) grid of tiles satisfying the horizontal and vertical constraints and with \( t_0 \) in the upper left position.

Show that TILING is NP complete.

[Hint: you can encode a Turing machine computation via such tiles. Let \( N \) be a nondeterministic TM. Consider a set of tiles that are labeled with the tape symbols from \( \Sigma \) plus an additional set of tiles that are labeled with triples \( \langle q, a, \delta \rangle \) where \( q \) is a state, \( a \in \Sigma \) and \( \delta \in \{0, 1\} \) indicates which nondeterministic choice the machine will make at this step, plus some additional tiles.

\[
\begin{array}{cccc}
\triangleright & q_0, a, \delta_0 & b; q & c & \square \\
\triangleright & d & q, b, \delta & c; q' & \square \\
\triangleright & d & e & q', c, \delta' & \square \\
\end{array}
\]

Suppose for example that choice \( \delta \) of \( N \) in state \( q \) looking at a \( b \) is to write \( e \), and go to the right into state \( q' \). Then the tile marked \( \langle q, b, \delta \rangle \) would have a signal at its bottom indicating that the next tile should contain symbol \( e \) and it could have a signal on its right indicating that the cell on its right should have a signal on its bottom indicating that the cell below should be in state \( q' \).]

3. Define the following series of sets in the Arithmetic Hierarchy:

\[
T_1 = \{n \mid \exists c \text{COMP}(n, n, n, c, 1)\} \\
T_2 = \{n \mid \exists x_1 \forall c \neg \text{COMP}(n, \langle n, x_1 \rangle, c, 1)\} \\
T_3 = \{n \mid \exists x_1 \forall x_2 \exists c \text{COMP}(n, \langle n, x_1, x_2 \rangle, c, 1)\} \\
T_{2k+1} = \{n \mid \exists x_1 \forall x_2 \exists x_3 \cdots \forall x_{2k} \exists c \text{COMP}(n, \langle n, x_1, x_2, \ldots, x_{2k} \rangle, c, 1)\} \\
T_{2k+2} = \{n \mid \exists x_1 \forall x_2 \exists x_3 \cdots \exists x_{2k+1} \forall c \neg \text{COMP}(n, \langle n, x_1, x_2, \ldots, x_{2k+1} \rangle, c, 1)\}
\]
Here we are using notation for $k$-tuples: $\langle x, y, z, \cdots \rangle \overset{\text{def}}{=} \langle x, \langle y, z, \cdots \rangle \rangle$, where $\langle x, y \rangle \overset{\text{def}}{=} P(x, y)$. The details of the encoding are not important; just that it is computable to encode and decode tuples.

COMP$(n, w, c, y)$ means that $M_n(w) = y$ and $c$ is a complete encoding of this computation.

Prove that for all $k$, $T_k$ is complete for $\Sigma_k$.

[Hint: don’t panic. Prove this first for $T_1$. Then prove it for $T_2$. If you want to quit there, you will get two thirds credit for this problem. The other third involves just keeping your cool and dealing with the notation.

Let

$$S = \{ n \mid (\exists x_1)(\forall x_2) \cdots (Q_k x_k) \varphi(n, x_1, \ldots, x_k) \},$$

be an arbitrary $\Sigma_k$ set, where $\varphi$ is a computable predicate.

Your job is to show that $S_k \leq T_k$. ]