1. This problem concerns the Arithmetic Hierarchy, which is at the top of the World-of-Computability-and-Complexity diagram. We say that a set of natural numbers, \( S \), is an element of \( \Sigma_k \) iff there is a ptime predicate \( \varphi \), such that,
\[
S = \{ n \mid \exists x_1 \forall x_2 \cdots Q_k x_k(\varphi(n, x_1, \ldots, x_k)) \},
\]
here \( Q_k \) is \( \forall \) if \( k \) is even and \( \exists \) if \( k \) is odd. Similarly, \( S \) is an element of \( \Pi_k \) iff,
\[
S = \{ n \mid \forall x_1 \exists x_2 \cdots Q'_k x_k(\psi(n, x_1, \ldots, x_k)) \},
\]
for some ptime predicate \( \psi \). Here \( Q'_k \) is \( \forall \) if \( k \) is odd and \( \exists \) if \( k \) is even.

(a) Prove that r.e. = \( \Sigma_1 \).
(b) Prove that co-r.e. = \( \Pi_1 \).

Define the Arithmetic Hierarchy to be \( \bigcup_{k=1}^{\infty} \Sigma_k \).
Classify the following sets by writing a formula that places them as low as you can in the arithmetic hierarchy. You do not have to prove that they cannot be placed in a lower class.

For example, \( \text{TOTAL} = \{ n \mid M_n \text{ halts on all inputs} \} \) is \( \Pi_2 \) because it can be written as
\[
\text{TOTAL} = \{ n \mid \forall x \exists z (\text{COMP}(n, x, L(z), R(z))) \}.
\]
(Here COMP\((n, x, c, y)\) is the ptime predicate meaning that \( c \) is a complete halting computation of TM \( M_n \) on input \( x \) and its output is \( y \).)

(c) \( \text{EMPTY} = \{ n \mid W_n = \emptyset \} \).
(d) \( \text{FINITE} = \{ n \mid W_n \text{ is finite} \} \).
(e) \( \text{PTM} = \{ M_i \mid \exists c \forall n (\text{TM } M_i \text{ runs in time } cn^c \text{ on all inputs of length } n) \} \).

2. Let \( \text{EMPTY-DFA} = \{ D \mid D \text{ a deterministic finite automaton and } L(D) = \emptyset \} \) and \( \text{EMPTY-NFA} = \{ N \mid N \text{ a nondeterministic finite automaton and } L(N) = \emptyset \} \).
Show that \( \text{EMPTY-DFA}, \text{EMPTY-NFA} \) and \( \text{EQUAL-DFA} \) are all NL complete.
3. Show that 2-SAT is NL-complete. I’d like you to do this using the following large hint:

Given a 2-CNF formula \( \varphi \), define the directed graph \( f(\varphi) = (V_\varphi, E_\varphi) \) as follows:

\[
V_\varphi = \{x_1, \overline{x_1}, \ldots, x_n, \overline{x_n}\}
\]
\[
E_\varphi = \{(u, v) \mid (\overline{u} \lor v) \text{ or } (v \lor \overline{u}) \text{ occurs in } \varphi.\}
\]

For example, for the formula \( \varphi \equiv (x \lor y) \land (\overline{y} \lor \overline{x}) \land (\overline{x} \lor z) \land (z \lor \overline{x}) \), the graph \( f(\varphi) \) is drawn below:

Recall that a directed graph is strongly connected iff for each pair of vertices \( v, w \) in the graph, there is a path from \( v \) to \( w \). A strongly connected component (SCC) of a directed graph is a maximal subgraph that is strongly connected.

The important observation about any 2CNF formula \( \varphi \) is the following:

\[
(\varphi \in \text{2-SAT}) \iff \forall x \in \text{VAR}(\varphi)("x, \overline{x} \text{ not in same SCC")}
\]

Using this observation, show that 2-SAT is NL complete. For extra credit, you may also prove the observation.

4. Prove that EMPTY-CFL is P-complete. First give an efficient algorithm for EMPTY-CFL. (It is possible to do this in linear time on a RAM.) For hardness, I suggest that you reduce MCVAL to EMPTY-CFL, where MCVAL is the monotone circuit-value problem which we will show in Lecture 8 is P-complete.