

**Hierarchy Theorems:** If  $f(n)$  is a  $\mathbf{C}$ -constructible function;

$\mathbf{C}$  is **DSPACE**, **NSPACE**, **DTIME**, or **NTIME**; and,

if  $g(n)$  is sufficiently smaller than  $f(n)$

**Then**  $\mathbf{C}[g(n)]$  is strictly contained in  $\mathbf{C}[f(n)]$ .

$g(n)$  **sufficiently smaller** than  $f(n)$  means:

$$\lim_{n \rightarrow \infty} \left( \frac{g(n)}{f(n)} \right) = 0$$

$\mathbf{C} = \mathbf{DSPACE}, \mathbf{NSPACE}$

$$\lim_{n \rightarrow \infty} \left( \frac{g(n+1)}{f(n)} \right) = 0$$

$\mathbf{C} = \mathbf{NTIME}$

$$\lim_{n \rightarrow \infty} \left( \frac{g(n) \log(g(n))}{f(n)} \right) = 0$$

$\mathbf{C} = \mathbf{DTIME}$

**Def:** Function  $f : \mathbf{N} \rightarrow \mathbf{N}$  is **C-constructible** if the map

$$1^n \mapsto f(n)$$

is computable in the complexity class  $\mathbf{C}[f(n)]$ .

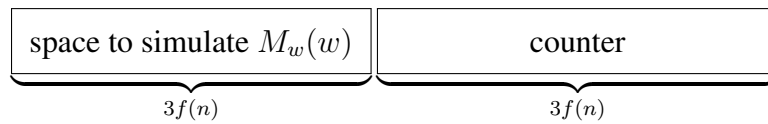
For example a function  $f(n)$  is **DSPACE**-constructible if the function  $f(n)$  can be deterministically computed from the input  $1^n$ , using space at most  $O[f(n)]$ .

**Fact:** All reasonable functions greater than or equal to  $\log n$  are **DSPACE**-constructible, and all reasonable functions greater than or equal to  $n$  are **DTIME**-constructible.

**Space Hierarchy Thm:** If  $f \geq \log n$  is space constructible and  $\lim_{n \rightarrow \infty} \left( \frac{g(n)}{f(n)} \right) = 0$ , Then  $DSPACE[g(n)] \subsetneq DSPACE[f(n)]$

**Proof:** Build  $DSPACE[f(n)]$  machine,  $D$ , on input:  $w, n = |w|$

1. Mark off  $6f(n)$  tape cells, ( $f$  space constructible)
2. Simulate  $M_w(w)$  using space  $3f(n)$ , time  $\leq 2^{3f(n)}$
3. **if** ( $M_w(w)$  needs more space or time) **then return(17)**
4. **else if** ( $M_w(w) = \text{accept}$ ) **then reject**
5. **else accept** // ( $M_w(w) = \text{reject}$ )



**Claim:**  $\mathcal{L}(D) \in \text{DSPACE}[f(n)] - \text{DSPACE}[g(n)]$

**Proof:**  $\mathcal{L}(D) \in \text{DSPACE}[f(n)]$  by construction.

**Suppose**  $\mathcal{L}(D) \in \text{DSPACE}[g(n)]$ .

Let  $\mathcal{L}(M_w) = \mathcal{L}(D)$ ,  $M_w$  uses  $cg(n)$  space.

Choose  $N$  s.t.  $\forall n > N (cg(n) < f(n))$ .

Choose  $w'$ ,  $M_{w'}(\cdot) = M_w(\cdot)$ ,  $|w'| > N$

On input  $w'$ ,  $D$  successfully simulates  $M_{w'}(w')$  in  $3f(n)$  space and  $2^{3f(n)}$  time.

$$\begin{aligned} w' \in \mathcal{L}(D) &\Leftrightarrow w' \notin \mathcal{L}(M_{w'}) \Leftrightarrow w' \notin \mathcal{L}(M_w) \Leftrightarrow w' \notin \mathcal{L}(D) \\ &\Rightarrow \Leftarrow \end{aligned}$$

□