Hierarchy Theorems: **If** $f(n)$ is a **C**-constructible function;

C is DSPACE, NSPACE, DTIME, or NTIME; and,

if $g(n)$ is sufficiently smaller than $f(n)$

Then $\mathbf{C}[g(n)]$ is strictly contained in $\mathbf{C}[f(n)]$.

$g(n)$ **sufficiently smaller** than $f(n)$ means:

$$\lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right)=0$$
$\mathbf{C}=$ DSPACE, NSPACE

$$\lim_{n\to\infty}\left(\frac{g(n+1)}{f(n)}\right)=0$$
$\mathbf{C}=$ NTIME

$$\lim_{n\to\infty}\left(\frac{g(n)\log(g(n))}{f(n)}\right)=0$$
$\mathbf{C}=$ DTIME

Def: Function $f : \mathbf{N} \to \mathbf{N}$ is **C-constructible** if the map

$$1^n \mapsto f(n)$$

is computable in the complexity class $\mathbf{C}[f(n)]$.

For example a function $f(n)$ is DSPACE-constructible if the function $f(n)$ can be deterministically computed from the input 1^n, using space at most $O[f(n)]$.

Fact: All reasonable functions greater than or equal to $\log n$ are DSPACE-constructible, and all reasonable functions greater than or equal to n are DTIME-constructible.

Space Hierarchy Thm: If $f \geq \log n$ is space constructible and $\lim_{n\to\infty}\left(\frac{g(n)}{f(n)}\right) = 0$, Then DSPACE$[g(n)] \subsetneq$ DSPACE[

Proof: Build DSPACE$[f(n)]$ machine, D, on input: w, $n = |w|$

1. Mark off $6f(n)$ tape cells, (f space constructible)
2. Simulate $M_w(w)$ using space $3f(n)$, time $\leq 2^{3f(n)}$
3. **if** ($M_w(w)$ needs more space or time) **then return**(17)
4. **else if** ($M_w(w)$ = **accept**) **then reject**
5. **else accept** // ($M_w(w)$ = **reject**)

$$\underbrace{\boxed{\text{space to simulate } M_w(w)}}_{3f(n)}\underbrace{\boxed{\qquad\qquad\text{counter}\qquad\qquad}}_{3f(n)}$$

Claim: $\mathcal{L}(D) \in$ DSPACE$[f(n)]$ $-$ DSPACE$[g(n)]$

Proof: $\mathcal{L}(D) \in$ DSPACE$[f(n)]$ by construction.

Suppose $\mathcal{L}(D) \in$ DSPACE$[g(n)]$.

Let $\mathcal{L}(M_w) = \mathcal{L}(D)$, M_w uses $cg(n)$ space.

Choose N s.t. $\forall n > N\ (cg(n) < f(n))$.

Choose w', $M_{w'}(\cdot) = M_w(\cdot)$, $|w'| > N$

On input w', D successfully simulates $M_{w'}(w')$ in $3f(n)$ space and $2^{3f(n)}$ time.

$$w' \in \mathcal{L}(D) \ \Leftrightarrow\ w' \notin \mathcal{L}(M_{w'}) \ \Leftrightarrow\ w' \notin \mathcal{L}(M_w) \ \Leftrightarrow\ w' \notin \mathcal{L}(D)$$

$$\Rightarrow\Leftarrow \qquad \square$$