Theorem 7.1 *Cook's Thm:* SAT *is* NP*-complete.*

Proof:

Let $B \in \mathrm{NP}$. By Fagin's theorem,

$$B \;=\; \{\mathcal{A} \mid \mathcal{A} \models \Phi\}; \qquad \Phi \;\equiv\; \exists C_0^{2k} \cdots C_{g-1}^{2k} \Delta^k \, \forall x_1 \cdots x_t \, (\alpha(\overline{x}))$$

with α quantifier-free and CNF,

$$\alpha(\bar{x}) \quad = \quad \bigwedge_{i=1}^{r} \bigvee_{j=1}^{s} \lambda_{i,j}(\bar{x})$$

where each $\lambda_{i,j}$ is a literal.

$\mathcal{A}$ arbitrary, $n = \|\mathcal{A}\|$, Define boolean formula $\varphi_{\mathcal{A}}$:

boolean variables: $C_i(e_1, \ldots, e_{2k}),\; \Delta(e_1, \ldots, e_k)$

$i = 1, \ldots, g,\; e_1, \ldots, e_{2k} \in |\mathcal{A}|$

literals: $\lambda_{i,j}(\bar{e})$, $i = 1, \ldots, r$, $j = 1, \ldots, s$, $\bar{e} \in |\mathcal{A}|^t$

$\lambda'_{i,j}(\bar{e})$ is $\lambda_{i,j}(\bar{e})$ with $R(\bar{e})$, replaced by $\top$ or $\perp$ according as $\mathcal{A} \models R(\bar{e})$; $C_i(\bar{e})$, and $\Delta(\bar{e})$ are just boolean variables.

$$\begin{aligned} \Phi \;&\equiv\; \exists C_0^{2k} \cdots C_{g-1}^{2k} \Delta^k \, \forall x_1 \cdots x_t \bigwedge_{i=1}^{r} \bigvee_{j=1}^{s} \lambda_{i,j}(\bar{x}) \\ \varphi(\mathcal{A}) \;&\equiv\; \bigwedge_{e_1,\ldots,e_t \in |\mathcal{A}|} \bigwedge_{i=1}^{r} \bigvee_{j=1}^{s} \lambda'_{i,j}(\bar{e}) \end{aligned}$$

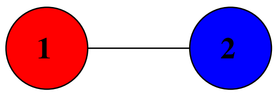
$$\mathcal{A} \in B \quad \Leftrightarrow \quad \mathcal{A} \models \Phi \quad \Leftrightarrow \quad \varphi(\mathcal{A}) \in \mathrm{SAT}$$

$\square$

Example: Fagin's Theorem implies Cook's Theorem

$$\begin{aligned}\Phi_{\text{2-color}} \quad \equiv \quad & \exists R^1\, \exists B^1\, \forall x, y\, \big[(R(x) \vee B(x)) \\ & \qquad \wedge \quad (\neg E(x,y) \vee \neg R(x) \vee \neg R(y)) \\ & \qquad \wedge \quad (\neg E(x,y) \vee \neg B(x) \vee \neg B(y))\big]\end{aligned}$$

G



boolean variables: r_1, r_2, b_1, b_2

$$\begin{aligned}\varphi_G \quad \equiv \quad & (r_1 \vee b_1) \wedge (\top \vee \overline{r_1} \vee \overline{r_1}) \wedge (\top \vee \overline{b_1} \vee \overline{b_1}) \\ & (r_1 \vee b_1) \wedge (\bot \vee \overline{r_1} \vee \overline{r_2}) \wedge (\bot \vee \overline{b_1} \vee \overline{b_2}) \\ & (r_2 \vee b_2) \wedge (\bot \vee \overline{r_2} \vee \overline{r_1}) \wedge (\bot \vee \overline{b_2} \vee \overline{b_1}) \\ & (r_2 \vee b_2) \wedge (\top \vee \overline{r_2} \vee \overline{r_2}) \wedge (\top \vee \overline{b_2} \vee \overline{b_2})\end{aligned}$$

Simplifies to: $(r_1 \vee b_1) \wedge (r_2 \vee b_2) \wedge (\overline{r_1} \vee \overline{r_2}) \wedge (\overline{b_1} \vee \overline{b_2})$