Despite Ladner's Theorem, there are very few natural problems that are:

- Known to be in NP, and
- Not known to be NP-complete, and
- Not known to be in P

Examples:

- Factoring natural numbers
- Graph Isomorphism
- Model Checking the μ-Calculus

$$\text{PRIME} \quad = \quad \left\{ m \in \mathbf{N} \;\middle|\; m \text{ is prime} \right\}$$

Prop: $\overline{\text{PRIME}} \in \text{NP}$

Proof:

$$\begin{aligned} m \in \overline{\text{PRIME}} \quad \Leftrightarrow \quad & m < 2 \quad \vee \\ & \exists xy \, (1 < x < m \; \wedge \; x \cdot y = m) \end{aligned}$$

$\square$

Question: Is PRIME $\in$ NP?

Fact 15.1 (Fermat's Little Thm) *Let p be prime and $0 < a < p$, then,* $a^{p-1} \equiv 1 \pmod{p}$.

$$\mathbf{Z}_n^\star \quad = \quad \left\{ a \in \{1, 2, \ldots, n-1\} \;\middle|\; \text{GCD}(a, n) = 1 \right\}$$

$Z_n^\star$ is the multiplicative group of integers mod n that are relatively prime to n.

Euler's phi function: $\varphi(n) \quad = \quad |\mathbf{Z}_n^\star|$

Prop: If $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime factorizaton of n, then

$$\varphi(n) = n(p_1 - 1)(p_2 - 1) \cdots (p_k - 1)/(p_1 p_2 \cdots p_k)$$

Euler's Thm: For any n and any $a \in \mathbf{Z}_n^\star$, $\quad a^{\varphi(n)} \equiv 1 \pmod{n}$.

Fact: Let $p > 2$ be prime. Then $\mathbf{Z}_p^\star$ is a cyclic group of order $p - 1$. That is,

$$\mathbf{Z}_p^\star = \{a, a^2, a^3, \ldots, a^{p-1}\}$$

$$m \in \text{PRIME} \quad \Leftrightarrow \quad \exists a \in \mathbf{Z}_m^\star \,(\text{ord}(a) = m - 1)$$

Pratt's Thm: PRIME $\in$ NP.

Proof: Given m,

1. Guess a, $1 < a < m$
2. Check $a^{m-1} \equiv 1 \pmod{m}$ by repeated squaring.
3. Guess prime factorization: $m - 1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$
4. Check for $1 \leq i \leq k,\ a^{m-1/p_i} \not\equiv 1(\mod m)$
5. Recursively check that $p_1, p_2, \ldots, p_k$ are prime.

Divide and Conquer NP Algorithm:

$$T(n) = O(n^2) + T(n-1)$$

$$T(n) = O(n^3)$$

□

Cor: PRIME and FACTORING are in NP $\cap$ co-NP.

Proof: PRIME: immediately from Pratt's Thm.

FACTORING is the problem of given N, find it's prime factorization: $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$.

Think of this as a decision problem by putting the factorization in a standard form, e.g., $p_1 < p_2 < \cdots < p_k$, and asking if bit i of the factorization is "1".

This is in NP $\cap$ co-NP because an NP or co-NP machine can guess the unique prime factorization, check that it is correct, and then read bit i. □

More Primality Testing

$a \in \mathbf{Z}_m^\star$ is a **quadratic residue** mod m iff, $\exists b\,(b^2 \equiv a \pmod{m})$

For p prime let,

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue mod } p \\ -1 & \text{otherwise} \end{cases}$$

Generalize to $\left(\frac{a}{m}\right)$ when m is not prime,

$$\left(\frac{a}{mn}\right) = \left(\frac{a}{m}\right)\left(\frac{a}{n}\right)$$

$$\left(\frac{a}{m}\right) = \left(\frac{a \,\%\, m}{m}\right)$$

Quadratic Reciprocity Thm: [Gauss] For odd a, m,

$$\left(\frac{a}{m}\right) = \begin{cases} \left(\frac{m}{a}\right) & \text{if } a \equiv 1 \pmod{4} \text{ or } m \equiv 1 \pmod{4} \\ -\left(\frac{m}{a}\right) & \text{if } a \equiv 3 \pmod{4} \text{ and } m \equiv 3 \pmod{4} \end{cases}$$

$$\left(\frac{2}{m}\right) = \begin{cases} 1 & \text{if } m \equiv 1 \pmod{8} \text{ or } m \equiv 7 \pmod{8} \\ -1 & \text{if } m \equiv 3 \pmod{8} \text{ or } m \equiv 5 \pmod{8} \end{cases}$$

Thus, we can calculate $\left(\frac{a}{m}\right)$ efficiently. For example,

$$\begin{aligned} \left(\frac{107}{351}\right) &= -\left(\frac{351}{107}\right) = -\left(\frac{30}{107}\right) \\ &= -\left(\frac{2}{107}\right)\left(\frac{15}{107}\right) = -\left(\frac{107}{15}\right) \\ &= -\left(\frac{2}{15}\right) = -1 \end{aligned}$$

$$107 \equiv 351 \equiv 15 \equiv 3 \text{ (mod 4)}$$

$$107 \equiv 3 \pmod{8}; \qquad 15 \equiv 7 \pmod{8}$$

Fact:[Gauss] For p prime, $a \in \mathbf{Z}_p^\star$, $\quad \left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$.

Fact: If m not prime then,

$$\left|\left\{a \in \mathbf{Z}_m^\star \;\middle|\; \left(\frac{a}{m}\right) \equiv a^{\frac{m-1}{2}} \pmod{m}\right\}\right| \;<\; \frac{m-1}{2}$$

Solovay-Strassen Primality Algorithm:

1. Input is odd number m
2. For $i := 1$ to k **do** {
3. choose $a < m$ at random
4. **if** GCD$(a, m) \neq 1$ **return**("not prime")
5. **if** $\left(\frac{a}{m}\right) \not\equiv a^{\frac{m-1}{2}} \pmod{m}$ **return**("not prime")
6. }
7. **return**("probably prime")

Thm:

- If m is prime then Solovay-Strassen(m) returns "probably prime".
- If m is not prime, then the probability that Solovay-Strassen(m) returns "probably prime" is less than $1/2^k$.

Cor: PRIME $\in$ "Truly Feasible"

Fact: [Agrawal, Kayal, and Saxena, 2002] PRIME $\in$ P

Def: A decision problem S is in BPP (Bounded Probabilistic Polynomial Time) iff there is a probabilistic, polynomial-time algorithm A such that for all inputs w,

$$\begin{aligned} &\textbf{if}\ \ (w \in S)\ \ \textbf{then}\ \ \text{Prob}(A(w) = 1) \geq \frac{2}{3} \\ &\textbf{if}\ \ (w \notin S)\ \ \textbf{then}\ \ \text{Prob}(A(w) = 1) \leq \frac{1}{3} \end{aligned}$$

Prop: If $S \in$ BPP then there is a probabilistic, polynomial-time algorithm A' such that for all n and all inputs w of length n,

$$\textbf{if}\ \ (w \in S)\ \ \textbf{then}\ \ \text{Prob}(A'(w) = 1) \geq 1 - \frac{1}{2^n}$$

$$\textbf{if}\ \ (w \notin S)\ \ \textbf{then}\ \ \text{Prob}(A'(w) = 1) \leq \frac{1}{2^n}$$

Proof: Iterate A polynomially many times and answer with the majority. Probability the mean is off by $\frac{1}{3}$ decreases exponentially with n — Chernoff bounds. $\square$

Is BPP equal to P???

Probably, because pseudo-random number generators are good.

Is randomness ever useful?

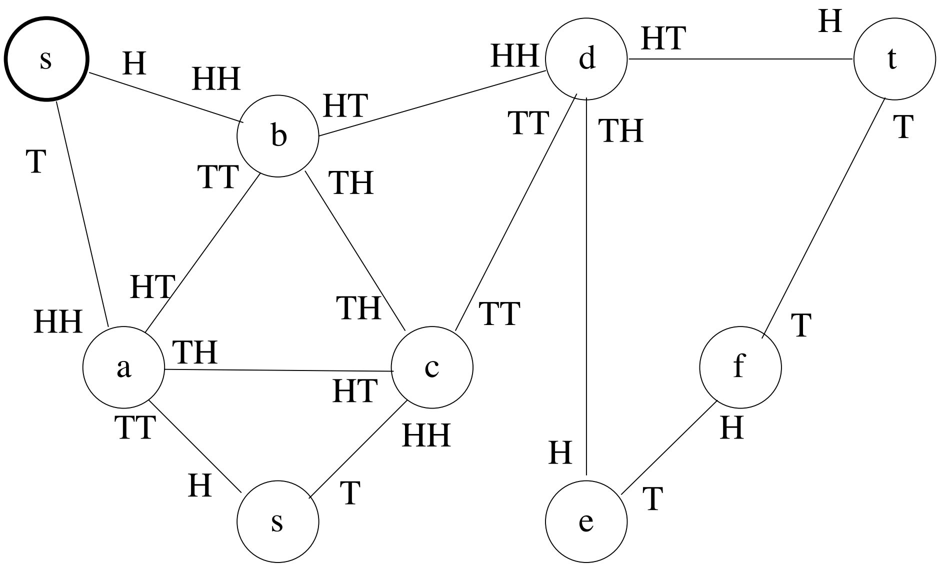
Yes: *Theory of Games and Economic Behavior,* by John Von Neumann, and Oskar Morgenstern, Princeton university press, 1944.

Colonel Kelly:

Which base to inspect?

If we randomize, then our opponent cannot know what we will do.

$$\mathrm{UREACH} \quad = \quad \left\{G, \text{ undirected} \;\middle|\; s \overset{\star}{\underset{G}{\to}} t\right\}$$



Fact 15.2 *Consider a random walk in a connected undirected graph G. Let $T(i)$ be the expected number of steps until we have reached all vertices, assuming we start at vertex i. Then,* $\quad T(i) \;\leq\; 2m(n-1)$, *where* $n = |V|$, $m = |E|$.

Corollary 15.3 $\quad$ UREACH $\in$ BPL.

Definition 15.4 A *universal traversal sequence* for graphs on n nodes, is a sequence of instructions, $q = a_1 a_2 a_3 \cdots a_t \in \{1, \ldots, n-1\}^\star$, such that for any **undirected** graph on n nodes, if we start at s in G and follow q, then we will visit every vertex in the connected component of s. □

Fact 15.5 *Undirected graphs with n vertices have universal traversal sequences of length $O(n^3)$.*

Fact 15.6 (Reingold, 2004) $\quad$ UREACH $\in$ L

Proof idea: derandomization of universal traversal sequences using expander graphs. □

Corollary 15.7 $\quad$ Symmetric-L $=$ L