# 1 Prop Logic Definitions and Abbreviations

$$\mathbf{P}_{\mathbf{var}} \quad \stackrel{\text{def}}{=} \quad \{p, q, r, s, p_0, q_0, r_0, s_0, p_1, q_1, r_1, s_1, \ldots\}$$ **Prop Logic Variables**

**Definition 1.1** [Inductive Definition of Propositional Formulas ($\mathbf{P}_{\mathbf{fmla}}$) (Syntax of Prop Logic)

**base 0.** $\top, \bot \in \mathbf{P}_{\mathbf{fmla}}$

**base 1.** If $a \in \mathbf{P}_{\mathbf{var}}$ then $a \in \mathbf{P}_{\mathbf{fmla}}$

**inductive 2.** If $\alpha \in \mathbf{P}_{\mathbf{fmla}}$ then $\neg\alpha \in \mathbf{P}_{\mathbf{fmla}}$

**inductive 3.** If $\alpha, \beta \in \mathbf{P}_{\mathbf{fmla}}$ then $(\alpha \vee \beta) \in \mathbf{P}_{\mathbf{fmla}}$

$\square$

## 1.1 Precedence

1. $\neg$ binds most tightly, then
2. $\wedge, \vee$, then
3. $\rightarrow, \leftrightarrow$
4. $\rightarrow$ associates as follows: $\quad \alpha \rightarrow \beta \rightarrow \gamma \quad \equiv \quad \alpha \rightarrow (\beta \rightarrow \gamma)$

Example: $\quad \neg\alpha \wedge \beta \rightarrow \neg\gamma \vee \delta \rightarrow \epsilon \quad \equiv \quad (\neg\alpha \wedge \beta) \rightarrow ((\neg\gamma \vee \delta) \rightarrow \epsilon)$

## 1.2 Semantics of Prop Logic

A **truth assignment** or **world** is a function $\mathcal{A} : D \rightarrow \{0, 1\}$ where $\mathrm{dom}(\mathcal{A}) = D \subseteq \mathbf{P}_{\mathbf{var}}$.
Extend $\mathcal{A}$ to $\mathcal{A}' : \mathbf{P}_{\mathbf{fmla}}(D) \rightarrow \{0, 1\}$ inductively as follows:

1. $\mathcal{A}(\top) \stackrel{\text{def}}{=} 1$; $\mathcal{A}(\bot) \stackrel{\text{def}}{=} 0$
2. for $v \in D$, $\mathcal{A}'(v) \stackrel{\text{def}}{=} \mathcal{A}(v)$
3. $\mathcal{A}'(\neg\alpha) \stackrel{\text{def}}{=} 1 - \mathcal{A}'(\alpha)$
4. $\mathcal{A}'(\alpha \vee \beta) \stackrel{\text{def}}{=} \max(\mathcal{A}'(\alpha), \mathcal{A}'(\beta))$

[For convenience, we will assume that $\mathcal{A} = \mathcal{A}'$, i.e., don't bother writing the " $'$ ".]

We say that $\mathcal{A}$ is **suitable** or **appropriate** for $\alpha$ if $\mathrm{var}(\alpha) \subseteq \mathrm{dom}(\mathcal{A})$.

Example: if $\mathrm{dom}(\mathcal{A}) = \{p, q\}$, then $\mathcal{A}$ is suitable for $p \vee \neg p \to \top$, but not for $\perp \to q \vee r$.

Note that a **Truth Table** can provide an equivalent definition for the semantics of PropLogic.

| truth assignment | $p$ | $q$ | $\neg p$ | $p \vee q$ | $p \wedge q$ | $p \to q$ |
|---|---|---|---|---|---|---|
| $\mathcal{A}_0$ | 0 | 0 | 1 | 0 | 0 | 1 |
| $\mathcal{A}_1$ | 0 | 1 | 1 | 1 | 0 | 1 |
| $\mathcal{A}_2$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $\mathcal{A}_3$ | 1 | 1 | 0 | 1 | 1 | 1 |

**Note:** If $\alpha \in \mathbf{P}_{\mathbf{fmla}}$ has $n$ variables, then there are $2^n$ possible truth assignments of interest. These are all the different worlds that might exist from $\alpha$'s point of view.

## 1.3 Abbreviations

**"$\hookrightarrow$" is an abbreviation for "is an abbreviation for"**

$$\begin{aligned}
\alpha \wedge \beta &\quad \hookrightarrow \quad \neg(\neg\alpha \vee \neg\beta) \\
\alpha \to \beta &\quad \hookrightarrow \quad \neg\alpha \vee \beta \\
a \leftrightarrow \beta &\quad \hookrightarrow \quad (a \to \beta) \wedge (\beta \to a)
\end{aligned}$$

(See **abbreviations.pdf** on the syllabus page:

`https://people.cs.umass.edu/~immerman/cs513/syllabus.html`

where I post important abbreviations.)