5.1 Syntax of First Order Logic with Equality

- \( \text{VAR} \overset{\text{def}}{=} \{x, y, z, u, w, x_0, y_0, \ldots, x_1, y_1, \ldots\} \)
- Vocabulary: \( \Sigma = (R_1^{a_1}, R_2^{a_2}, \ldots, R_s^{a_s}; f_1^{r_1}, f_2^{r_2}, \ldots, f_t^{r_t}) \)
- Relation Symbols: \( R_k^i \) where \( k \geq 0 \) and \( i \geq 1 \)
- Function Symbols: \( f_k^i \) where \( k \geq 0 \) and \( i \geq 1 \) (typically abbreviated as just \( f, g, h \))
- Constant Symbols: \( f_0^i \) (typically abbreviated as just \( a, b, c, d, k \))

**Definition 5.1** \( \text{term}(\Sigma) \)

**Base Case:** If \( v \in \text{VAR} \) then \( v \in \text{term}(\Sigma) \).

**Inductive Case:** If \( t_1, t_2, \ldots, t_r \in \text{term}(\Sigma) \) and \( f^r \in \Sigma \) then \( f(t_1, \ldots, t_r) \in \text{term}(\Sigma) \)

A term \( t \in \text{term}(\Sigma) \) is a syntactic object that any structure \( A \in \text{STRUC}[\Sigma] \) will have to interpret as an element \( t^A \in |A| \).

**Definition 5.2** \( \mathcal{L}(\Sigma) \) (First Order formulas of Vocab \( \Sigma \))

**Base Case:** atomic formulas

If \( t_1, \ldots, t_a \in \text{term}(\Sigma) \) and \( R^a \in \Sigma \) then \( R(t_1, \ldots, t_a) \in \mathcal{L}(\Sigma) \).

**Inductive Steps:**

If \( \alpha, \beta \in \mathcal{L}(\Sigma) \) and \( v \in \text{VAR} \) then

1. \( \neg \alpha \in \mathcal{L}(\Sigma) \)
2. \( (\alpha \lor \beta) \in \mathcal{L}(\Sigma) \)
3. \( \exists v(\alpha) \in \mathcal{L}(\Sigma) \)

**Abbreviation:** \( \forall x(\alpha) \leftrightarrow \neg \exists x(\neg \alpha) \)
5.2 Semantics of First Order Logic with Equality

Definition 5.3. $\mathcal{A}$ is a logical structure of vocabulary $\Sigma$ ($\mathcal{A} \in \text{STRUC}[\Sigma]$) iff

$$\mathcal{A} = (|A|, R_1^A, \ldots, R_s^A; f_1^A, \ldots, f_t^A)$$

$|A| \neq \emptyset$,

$R_i^A \subseteq |A|^{a_i}$, $\mathcal{A}$ interprets $P_i$ as an $a_i$-ary relation over its universe.

$f_i^A : |A|^{r_i} \to |A|$, $\mathcal{A}$ interprets $f_i$ as a total function taking $r_i$ arguments.

□

5.3 Tarski’s Definition of Truth

Definition 5.4. Every structure $\mathcal{A} \in \text{STRUC}[\Sigma]$ interprets every term $t \in \text{term}(\Sigma)$ as an element, $t^A$ of its universe.

**base case:** If $v \in \text{VAR}$ then $v^A \in |A|$, i.e., $\mathcal{A}$ gives a default value $v^A$ to every variable, $v$.

**inductive case:** If $t_1, \ldots, t_r$ are terms already defined by $\mathcal{A}$, and $f^r \in \Sigma$, then

$$f(t_1, \ldots, t_r)^A \overset{\text{def}}{=} f^A(t_1^A, \ldots, t_r^A)$$

□

Definition 5.5. [Truth] Let $\varphi \in \mathcal{L}(\Sigma)$, $\mathcal{A} \in \text{STRUC}[\Sigma]$. We inductively define whether or not $\mathcal{A} \models \varphi$.

**Notation:** For $\alpha \in \text{term}(\Sigma)$ and $\mathcal{A} \in \text{STRUC}[\Sigma]$, $\mathcal{A}(\alpha) = 1$ iff $\mathcal{A} \models \alpha$, i.e., $\alpha$ is true in $\mathcal{A}$. A structure $\mathcal{A}$ that satisfies a formula $\alpha$ is called a **model** of $\alpha$. $\mathcal{A}(\alpha) = 0$ iff $\mathcal{A} \not\models \alpha$, i.e., $\alpha$ is false in $\mathcal{A}$.

**Base Case:** (Atomic Formulas)

1. $\mathcal{A} \models R_i(t_1, \ldots, t_{a_i})$ iff $(t_1^A, \ldots, t_{a_i}^A) \in R_i^A$ (Gizem figured out what this definition had to be.)

2. $\mathcal{A} \models t_1 = t_2$ iff $t_1^A = t_2^A$, i.e., we insist that the binary predicate symbol, “$=$”, is always interpreted as “true equality”, i.e., $t_1^A$ and $t_2^A$ are the exact same element of $|A|$. (Mike figured out this definition.) Put another way, $=^A = \{(a, a) \mid a \in |A|\}$.

**Inductive Cases:**

1. $\mathcal{A} \models \neg \alpha$ iff $\mathcal{A} \not\models \alpha$

2. $\mathcal{A} \models (\alpha \vee \beta)$ iff $\mathcal{A} \models \alpha$ or $\mathcal{A} \models \beta$
3. \( A \models \exists v(\alpha) \iff \text{there exists } a \in |A| \text{ such that } A[a/v] \models \alpha \)

where \( A[a/v] \) is defined to be the exact same structure as \( A \) with the single exception that the default value of \( v \) in \( A[a/v] \) is \( a \), i.e., \( v^{A[a/v]} = a \).

\[ \square \]

### 5.4 Examples

Some vocabularies:

- \( \Sigma_{\text{graph}} = (E^2;) \)
- \( \Sigma_{\text{st-graph}} = (E^2; s, t) \)
- \( \Sigma_N = (\leq^2[\text{infix}]; 0, \text{Suc}^1, +^2[\text{infix}], \cdot^2[\text{infix}]) \)
- \( \Sigma_{\text{set}} = (\in^2[\text{infix}]; \emptyset) \)
- \( \Sigma_{\text{group}} = (; \circ^2[\text{infix}], e) \)

Some structures:

\[ G_0 \]

\[ G_0' \]

\[ s^{G_0} = 0, t^{G_0} = 1 \]

\[ E^{G_0} = \{(0, 1), (2, 2)\} \]

\( G_0 \in \text{STRUC}[\Sigma_{\text{graph}}]; \ G_0' \in \text{STRUC}[\Sigma_{\text{st-graph}}]; \ |G_0| = |G_0'| = \{0, 1, 2\} \)

\( E(s, t) \in \mathcal{L}(\Sigma_{\text{st-graph}}); \ G_0' \models E(s, t) \)

Let \( N = \{0, 1, \ldots\}, \leq^N, \text{Suc}^N, +^N, \cdot^N \). \( N \) is the **standard model of the natural numbers**. \( N \in \text{STRUC}[\Sigma_N] \).

\[ \leq^N = \{(0, 0), (0, 1), \ldots, (1, 1), (1, 2), \ldots, (8, 9), (8, 10), \ldots\} \]

\[ \text{Suc}^N = \{(0, 1), (1, 2), (2, 3), \ldots\} \]

\[ +^N = \{(0, 0), (0, 1), \ldots, (2, 2), 4\ldots, (8, 9), 17\ldots\} \]

\[ \cdot^N = \{(0, 0), (0, 1), 0\ldots, (2, 2), 4\ldots, (8, 9), 72\ldots\} \]
Let $\alpha \equiv \forall xy(x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$. $\alpha$ is the “axiom of extensionality”, the first axiom of ZFC (Zermelo-Fraenkel plus Choice). It says, “Two sets are equal iff they have exactly the same elements.”

A group, $G$, is a non-empty set with a binary operation that is associative, has an identity and inverses.

The group theory axioms consist of $\gamma_1 \wedge \gamma_2 \wedge \gamma_3$:

- **Associative**: $\gamma_1 \equiv \forall xyz (x \circ y) \circ z = x \circ (y \circ z)$
- **Identity**: $\gamma_2 \equiv \forall x (x \circ e) = x$
- **Inverse**: $\gamma_3 \equiv \forall x \exists y (x \circ y) = e$

A **group** is neither more nor less than a model of the group theory axioms.

A **graph** is neither more nor less than a structure of vocabulary $\Sigma_{\text{graph}}$.

Let $\psi \in L(\Sigma_{\text{graph}})$ say “loop-free and undirected”:

$$\psi \equiv \forall xy(\neg E(x, x) \wedge (E(x, y) \rightarrow E(y, x))) .$$

A **loop-free, undirected graph** is neither more not less than a model of $\psi$. 