3.1 CNF and DNF

**Conjunctive Normal Form (CNF):** “And of Or of literals”, e.g.,

\[ \varphi \equiv (\neg p \land \neg q \land r) \lor (\neg p \land q \land \neg r) \lor (p \land \neg q \land \neg r) . \]

**Disjunctive Normal Form (DNF):** “Or of And of literals”, e.g.,

\[ \neg \varphi \equiv (\neg p_1 \land p_2 \land \neg p_3) \lor (p_1 \land \neg p_3 \land p_4) \lor (\neg p_2 \land p_3 \land p_4) . \]

**Proposition 3.1** \( \forall \varphi \in \mathcal{P}_{\text{fmla}} \exists \alpha \beta \in \mathcal{P}_{\text{fmla}} \text{ such that } \varphi \equiv \alpha \equiv \beta, \alpha \in \text{CNF}, \beta \in \text{DNF}. \)

**Proof:** An exhaustive proof recursively uses the de Morgan laws and the distributive laws.

A different proof is that from the truth table of \( \varphi \), we can read off the DNF form of \( \varphi \) as the disjunction of the rows where \( \varphi \) is true. For example,

<table>
<thead>
<tr>
<th>( A )</th>
<th>( p )</th>
<th>( q )</th>
<th>( r )</th>
<th>( \varphi )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_0 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( A_1 )</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>( A_5 )</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( A_6 )</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( A_7 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Then to put \( \varphi \) in CNF, we can first put \( \neg \varphi \) in DNF and then negate both sides and use de Morgan’s law. \( \square \)

3.2 The Compactness Theorem for Propositional Logic

**Definition 3.2** Recall that \( \Gamma \) is satisfiable iff \( \exists \) an appropriate truth assignment \( \mathcal{A} \) such that \( \mathcal{A} \models \Gamma \), i.e., \( \forall \gamma \in \Gamma, \mathcal{A} \models \gamma \). \( \Gamma \) is finitely satisfiable iff for all finite subsets \( S \subseteq \Gamma \), \( S \) is satisfiable. \( \square \)

**Theorem 3.3 (Compactness Thm for Prop Logic)** For any set \( \Gamma \subseteq \mathcal{P}_{\text{fmla}} \), if \( \Gamma \) is finitely satisfiable, then \( \Gamma \) is satisfiable.

To prove Thm 3.3, we first prove the following:
Lemma 3.4 (König’s Infinity Lemma) Let $T$ be a binary tree, directed down from the root, and suppose that $T$ has infinitely many nodes. Then $T$ has an infinite branch, i.e., an infinite path down from the root.

Proof:
Start at the root node, $r$. Since $r$ has at most two children, and $r$ has infinitely many descendants, at least one of $r$’s children must have infinitely many descendents as well. Choose a child, $r_1$, having infinitely many descendents and repeat, i.e., one of $r_1$’s children has infinitely many descendents, call it $r_2$, and we continue this way forever. Thus we have constructed an infinite branch.

Proof: (of Theorem 3.3) We will use Lemma 3.4. Assume that $\text{var}(\Gamma) = \{p_1, p_2, \ldots\}$.

Let $D_i = \{\gamma \in \Gamma \mid \text{var}(\gamma) \subseteq \{p_1, \ldots, p_i\}\}$. We know that there are only finitely many different formulas (up to equivalence) on a fixed set of variables. Thus, we can remove extra copies of equivalent formulas and assume that each $D_i$ is finite. Since $\Gamma$ is finitely satisfiable, it follows that every $D_i$ is satisfiable.

Let $T = (V_T, E_T)$ where $V_T = \{\mathcal{A} \mid \exists i \in \mathbb{N} (\text{dom}(\mathcal{A}) = \{p_1, \ldots, p_i\} \& \mathcal{A} \models D_i)\}$ Note that $T$ has infinitely many nodes. We can think of $T$ as a binary tree as follows: The root is the empty truth assignment, $\mathcal{A}_\epsilon$, i.e., $\text{dom}(\mathcal{A}_\epsilon) = \emptyset$. $\mathcal{A}_\epsilon$ has two possible children: $\mathcal{A}_0, \mathcal{A}_1$, where $\text{dom}(\mathcal{A}_0) = \text{dom}(\mathcal{A}_1) = \{p_1\}$ and $\mathcal{A}_0(p_1) = 0, \mathcal{A}_1(p_1) = 1$. Note that since $D_1$ is satisfiable, at least one of $\mathcal{A}_0, \mathcal{A}_1$ is in $V_T$. In general, for $w \in \{0, 1\}^k$, a binary string of length $k$, $\text{dom}(\mathcal{A}_w) = \{p_1, \ldots, p_k\}$. $\mathcal{A}_w$ has two possible children, $\mathcal{A}_{w0}, \mathcal{A}_{w1}$ both with domain $\{p_1, \ldots, p_{k+1}\}$ and $\mathcal{A}_{w0}(p_{k+1}) = 0, \mathcal{A}_{w1}(p_{k+1}) = 1$. That is,

$$E_T = \{(\mathcal{A}_w, \mathcal{A}_{wi}) \mid \mathcal{A}_{wi} \in V_T \& i \in \{0, 1\}\}.$$

Thus, $T$ is an infinite binary tree. It is infinite because each $D_i$ is satisfiable, so for all $i \in \mathbb{N}$, $T$ has at least one node of depth $i$. By König’s Infinity Lemma, $T$ has an infinite branch,

$$B = \mathcal{A}_\epsilon, \mathcal{A}_{w0}, \mathcal{A}_{w1}, \ldots$$

Observe that $\mathcal{A}_\epsilon \subseteq \mathcal{A}_{w0} \subseteq \mathcal{A}_{w1} \cdots \subseteq \cdots$. Let $\mathcal{A} = \bigcup_{k=0}^{\infty} \mathcal{A}_{wk}$.

Claim: $\mathcal{A} \models \Gamma$.

To see this, we have to show that for all $\gamma \in \Gamma$, $\mathcal{A} \models \gamma$. Let $\gamma$ be an arbitrary element of $\Gamma$. Since $\gamma \in \mathbb{P}_{\text{fmla}}$ and formulas are finite, $\gamma$ has only finitely many propositional variables. Let $i$ be the max index of the propositional variables occuring in $\gamma$. Thus, $\gamma \in D_i$. Since $\mathcal{A} \models D_i$, it follows that $\mathcal{A} \models \gamma$. □