

König's Infinity Lemma: Let T be a binary tree with infinitely many nodes. Then T has an infinite branch.

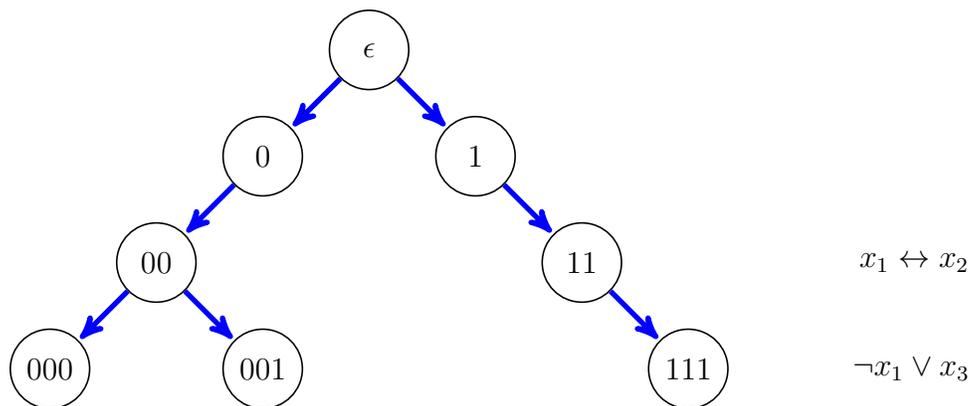
Proof: Consider such a T . We construct an infinite path by induction.

base case: Let $n_0 = \epsilon$, the root. Since T has infinitely many nodes, n_0 has infinitely many descendants.

inductive case: Assume inductively that we have a path $n_0 n_1 \dots n_i$ and n_i has infinitely many descendants. If n_i has only one child, call it $n_i b$, then let $n_{i+1} = n_i b$ and it has infinitely many descendants.

If n_i has both children, then since $(desc)(n_i) = \{n_i\} \cup (desc)(n_i 0) \cup (desc)(n_i 1)$, at least one of its children, $n_i b$, has infinitely many descendants. Let $n_{i+1} = n_i b$.

Thus, by induction, $\forall i$ we have a path $n_0 n_1 \dots n_i$ and n_i has infinitely many descendants. Thus $B = n_0 n_1 \dots$ is the desired infinite branch. \square



We say that a set of formulas M is **finitely sat** iff every finite subset of M is satisfiable.

PropCalc Compactness Thm: If M is a finitely sat set of propositional formulas, then M is satisfiable.

Proof: We may assume that no two formulas of M are equivalent. Define the following finite subsets of M :

$$D_i = \{\alpha \in M \mid \text{prop}(\mathbf{M}) \subseteq \{x_1, \dots, x_i\}\} \quad \text{Note: } D_1 \subseteq D_2 \subseteq D_3 \dots; \quad \bigcup_{i=1}^{\infty} D_i = M$$

Let $T = \{\mathcal{A} \mid \exists i \text{ dom}(\mathcal{A}) = \{x_1, \dots, x_i\}, \mathcal{A} \models D_i\}$.

For example, in the diagram above, $M = \{x_1 \leftrightarrow x_2, \neg x_1 \vee x_3, \dots\}$, $D_0 = D_1 = \emptyset$, $D_2 = \{x_1 \leftrightarrow x_2\}$, $D_3 = \{\neg x_1 \vee x_3\} \cup D_2$.

Note that T is an infinite binary tree since there are infinitely many D_i and they are all satisfiable. Thus, by König's Infinity Lemma, T has an infinite branch, which thus satisfies all of M . \square