Problems

1. Prove: $\mathcal{A} \cong \mathcal{B} \;\Rightarrow\; \mathcal{A} \equiv \mathcal{B}$

   Recall some definitions: let $\mathcal{A}, \mathcal{B}$ be structures of the same vocabulary, τ. We say that $\mathcal{A}$ is **isomorphic** to $\mathcal{B}$, in symbols $\mathcal{A} \cong \mathcal{B}$, iff there is a 1:1 and onto function $\eta : |\mathcal{A}| \to |\mathcal{B}|$ with the following properties:

   for every function symbol $f^r \in \tau$, and all elements $e_1, \dots e_r \in |\mathcal{A}|$,

   $$\eta(f^{\mathcal{A}}(e_1, \dots, e_r)) \quad = \quad f^{\mathcal{B}}(\eta(e_1), \dots, \eta(e_r)),$$

   and for every relation symbol $R^a \in \tau$, and all elements $e_1, \dots e_a \in |\mathcal{A}|$,

   $$\langle e_1, \dots, e_a \rangle \in R^{\mathcal{A}} \quad \Leftrightarrow \quad \langle \eta(e_1), \dots, \eta(e_a) \rangle \in R^{\mathcal{B}} \,.$$

   The map η is called an **isomorphism** from $\mathcal{A}$ to $\mathcal{B}$.

   We say that $\mathcal{A}$ is **elementarily equivalent** to $\mathcal{B}$, in symbols $\mathcal{A} \equiv \mathcal{B}$, iff for all first-order sentences φ of vocabulary τ, $\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi$.

   Note that the definition of isomorphism says that η honors all symbols of τ. Thus $\mathcal{A}$ and $\mathcal{B}$ are identical except for the names of the elements of the universe. Thus it should be no surprise that two isomorphic structures satisfy exactly the same sentences.

   To do this I suggest that you show by induction on φ that $\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi$. To do this by induction you will have to prove it not just for sentences but for all formulas. For this, you should make the added assumption that η preserves the default value of all variables, i.e., for all $x_i \in$ var,

   $$\eta(x_i^{\mathcal{A}}) \quad = \quad x_i^{\mathcal{B}}$$

   Note that since the default values of the variables do not affect the truth of sentences, you can just make the default values to meet the above condition.

2. Show directly that the following two notions are equivalent concerning a decision problem $S \subseteq \mathbf{N}$:

   Recognizable by a Turing Machine: $S = \mathcal{L}(M)$ for some TM M. Recall the definition

   $\mathcal{L}(M) \stackrel{\text{def}}{=} \{i \mid M(i) = 1\}$, i.e., $\mathcal{L}(M)$ is the set of inputs on which M eventually halts and outputs "1".

   Recursively Enumerable (r.e.): $S = \emptyset$ or $S = \{f(0), f(1), f(2), \dots\}$ for some total recursive function f. Recall that a total recursive function is a function computed by a TM which halts and answers on every input.

3. Which of the following pairs of CTL formulas are equivalent? For each pair, give an informal proof of equivalence or a Kripke structure on which they differ.

   (a) $\mathbf{EF}\varphi \vee \mathbf{EF}\psi$; $\quad \mathbf{EF}(\varphi \vee \psi)$

   (b) $\mathbf{AF}\varphi \vee \mathbf{AF}\psi$; $\quad \mathbf{AF}(\varphi \vee \psi)$

   (c) $\mathbf{A}[p\mathbf{U}\mathbf{A}[q\mathbf{U}r]]$; $\quad \mathbf{A}[\mathbf{A}[p\mathbf{U}q]\mathbf{U}r]$. [Hint: first think about a model that has only one path.]

4. Define the "release" operator as follows:

   $\alpha\mathbf{R}\beta \;\equiv\; \mathbf{G}\beta \;\vee\; [\beta\mathbf{U}(\alpha \wedge \beta)]$

   Intuitively, $\alpha\mathbf{R}\beta$, read, "α releases β", means that α being true releases β from the obligation of being true in the future, i.e, β must be true now and continue to be true in the future unless at some point α becomes true. After that, β need not be true anymore.

   Argue that **R** and **U** are dual in the sense that:

   $\alpha\mathbf{R}\beta \;\equiv\; \neg(\neg\alpha\mathbf{U}\neg\beta)$ and

   $\alpha\mathbf{U}\beta \;\equiv\; \neg(\neg\alpha\mathbf{R}\neg\beta)$