CS513:	**Homework 3**	Due at gradescope by midnight, 10/19/17

Reading Please finish reading Chapter 2 by the time this assignment is due.

I can't accept late hw's for Hw3: Because of the midterm coming up on Oct. 23, I will post solutions to hw3 at 10 a.m. on Friday, Oct. 20. We won't accept any late hw3 submissions after that.

Problems

1. [10 pts.] Write a formula, φ, in first-order logic with equality that is satisfiable, but only has infinite models, i.e., the universe must be infinite. Produce an infinite model for φ and argue why φ cannot have a finite model. [Hint: you can do this in a language with a single unary function symbol, f. You want to find a property of f which is only possible for infinite universes.]

2. [20 pts.] Use resolution to prove that $\exists x \forall y P(x, y) \vdash \forall y \exists x P(x, y)$ and show that the converse is not true by producing a structure that satisfies the second sentence but not the first.

3. [30 pts.] Do Exercise 85, p. 96. Assume that the whole universe consists of dragons. Use the predicates: $H(x)$, $G(y)$, $F(z)$, $C(v, w)$ to mean x is happy, y is green, z can fly, and v is a child of w, respectively. More explicitly:

   (a) Write in FO, HCF, meaning "Every dragon is happy if all its children can fly."

   (b) Convert to HCF_S, an equi-satisfiable Skolem formula in RPF-CNF, i.e., in RPF, with the quantifier-free part in CNF.

   (c) Write in FO, GF, meaning "Green dragons can fly."

   (d) Convert to GF_S, an equi-satisfiable Skolem formula in RPF-CNF.

   (e) Write in FO, GCG, meaning "A dragon is green if it is a child of at least one green dragon."

   (f) Convert to GCG_S, an equi-satisfiable Skolem formula in RPF-CNF.

   (g) Write in FO, GH, meaning "All green dragons are happy".

   (h) Convert $\neg$GH to NGH_S, an equi-satisfiable Skolem formula in RPF-CNF.

   (i) Using resolution, prove that HCF $\wedge$ GF $\wedge$ GCG $\vdash$ GH.

4. [25 pts.] Let $\mathcal{A}, \mathcal{A}'$ be logical structures of the same vocabulary. We say that $\mathcal{A}'$ is a **substructure** of $\mathcal{A}$ ($\mathcal{A}' \leq \mathcal{A}$) iff $|\mathcal{A}'| \subseteq |\mathcal{A}|$ and $\mathcal{A}'$ interprets all predicate symbols and function symbols the same way that $\mathcal{A}$ does, i.e., for each predicate symbol P of arity i, $P^{\mathcal{A}'} = P^{\mathcal{A}} \cap |\mathcal{A}'|^i$, and for each function symbol f of arity i, and for each $e_1, \ldots e_i \in |\mathcal{A}'|$, $f^{\mathcal{A}'}(e_1, \ldots, e_i) = f^{\mathcal{A}}(e_1, \ldots, e_i)$. Note in particular that since A' is a structure it must be closed under all defined functions. Example: for graphs, $G' \leq G$ iff G' is an induced subgraph of G, i.e., G' has a subset of G's vertices and all the edges between them that G has.

   A first-order formula is **universal** if it is in prenex normal form and all of its quantifiers are $\forall$'s. Similarly it is **existential** if it is in prenex normal form and all of its quantifiers are $\exists$'s. Suppose that $\mathcal{A} \leq \mathcal{B}$, i.e. $\mathcal{A}$ is a substructure of $\mathcal{B}$. Prove the following:

   (a) If φ is universal and $\mathcal{B} \models \varphi$ then $\mathcal{A} \models \varphi$.

   (b) If φ is existential and $\mathcal{A} \models \varphi$ then $\mathcal{B} \models \varphi$.

[Hint: (a) and (b) imply each other. Just prove one and show why the other follows. Your proof should be by induction on the number of quantifiers. For the base case, α is quantifier free. I suggest you prove by induction on α that $\mathcal{A} \models \alpha \Leftrightarrow \mathcal{B} \models \alpha$. You may assume that the two structures interpret the same free variables the same way, i.e., if one of them interprets x then they both do and $x^{\mathcal{A}} = x^{\mathcal{B}}$.]

5. [15 pts.] Do exercise 6.5 in the EF Games handout: `https://people.cs.umass.edu/~immerman/cs513/EFgames.pdf`. [Hint: there are three properties to prove, namely that $\sim_m^k$ is reflexive, symmetric and transitive. The transitive case is most interesting. For the symmetric and transitive cases, you are given the fact that Delilah has a winning strategy for certain game(s), and you must show that she has a winning strategy for some other game.