CS513:	**Crib Sheet for Midterm**	Fall, 2021

Def of PropCalc Formulas ($\mathbf{P_{fmla}}$) $\quad \mathbf{P_{var}} \stackrel{\text{def}}{=} \{p, q, r, s, p_0, q_0, r_0, s_0, p_1, q_1, r_1, s_1, \ldots\}$

base: $\top, \bot \in \mathbf{P_{fmla}}$; if $a \in \mathbf{P_{var}}$ then $a \in \mathbf{P_{fmla}}$; **ind:** If $\alpha, \beta \in \mathbf{P_{fmla}}$ then $\neg\alpha, (\alpha \vee \beta) \in \mathbf{P_{fmla}}$

Completeness Thms for PropCalc and FO: For Γ a set of formulas and φ a formula, $(\Gamma \models \varphi) \Leftrightarrow (\Gamma \vdash \varphi)$.

Compactness Thms: For Γ a set of formulas, (Γ is satisfiable) $\Leftrightarrow$ (Every finite subset of Γ is satisfiable).

FO Syntax: term(Σ) : VAR $\stackrel{\text{def}}{=} \{x, y, z, u, v, w, x_0, y_0, \ldots, x_1, y_1, \ldots\}$

base: If $v \in$ VAR then $v \in$ term(Σ);

ind: If $t_1, t_2, \ldots, t_r \in$ term(Σ); $f \in \Sigma$, ar$(f) = r$ then $f(t_1, \ldots, t_r) \in$ term(Σ)

FO formulas: $\mathcal{L}(\Sigma)$ **base:** If $t_1, \ldots, t_a \in$ term(Σ), $P \in \Sigma$, ar$(f) = a$ then $P(t_1, \ldots, t_a), t_1 = t_2 \in \mathcal{L}(\Sigma)$.

ind: If $\alpha, \beta \in \mathcal{L}(\Sigma)$ and $v \in$ VAR then $\neg\alpha$, $(\alpha \vee \beta)$, $\exists v(\alpha) \in \mathcal{L}(\Sigma)$

Tarski's Definition of Truth: $\mathcal{A} \in$ STRUC[Σ]

terms: base: If $v \in$ VAR then $v^{\mathcal{A}} \in |\mathcal{A}|$ **ind:** $t_1, \ldots, t_r \in$ term(Σ); $f \in \Sigma$, $f(t_1, \ldots, t_r)^{\mathcal{A}} \stackrel{\text{def}}{=} f^{\mathcal{A}}(t_1^{\mathcal{A}}, \ldots, t_r^{\mathcal{A}})$

atomic fmla: $\mathcal{A} \models P(t_1, \ldots, t_a)$ iff $(t_1^{\mathcal{A}}, \ldots, t_{a_i}^{\mathcal{A}}) \in P^{\mathcal{A}}$; $\mathcal{A} \models t_1 = t_2$ iff $t_1^{\mathcal{A}} = t_2^{\mathcal{A}}$

ind: $\mathcal{A} \models \neg\alpha$ iff $\mathcal{A} \not\models \alpha$; $\mathcal{A} \models (\alpha \vee \beta)$ iff $\mathcal{A} \models \alpha$ or $\mathcal{A} \models \beta$; $\mathcal{A} \models \exists v(\alpha)$ iff exists $a \in |\mathcal{A}|$, $\mathcal{A}[v/a] \models \alpha$

Truth Game: $\mathcal{A} \models \varphi$ iff Dumbledore wins on $(\mathcal{A}, \varphi)$; $\mathcal{A} \models \neg\varphi$ iff Gandalf wins on $(\mathcal{A}, \varphi)$

$\varphi = \forall x(\alpha)$	**G** picks $a \in \lvert\mathcal{A}\rvert$; proceed on $(\mathcal{A}[a/x], \alpha)$.	$\varphi = \exists x(\alpha)$	**D** picks $a \in \lvert\mathcal{A}\rvert$; proceed on $(\mathcal{A}[a/x], \alpha)$.
$\varphi = (\alpha \wedge \beta)$	**G** picks $\gamma \in \{\alpha, \beta\}$; proceed on $(\mathcal{A}, \gamma)$.	$\varphi = (\alpha \vee \beta)$	**D** picks $\gamma \in \{\alpha, \beta\}$; proceed on $(\mathcal{A}, \gamma)$.
φ is a literal	**D** wins iff $\mathcal{A} \models \varphi$.	φ is a literal	**G** wins iff $\mathcal{A} \models \neg\varphi$.

Convert FO Fmla to Equivalent Fmla in Rectified Prenex Normal (RPF) Form; then Skolemize

1. Remove all "$\rightarrow$"s using the fact that $\alpha \rightarrow \beta \;\equiv\; \neg\alpha \vee \beta$.
2. Push all "$\neg$"s all the way inside using de Morgan and quantifier rules:

   $\neg(\alpha \wedge \beta) \;\equiv\; \neg\alpha \vee \neg\beta$; $\neg(\alpha \vee \beta) \;\equiv\; \neg\alpha \wedge \neg\beta$; $\neg\forall x(\varphi) \;\equiv\; \exists x(\neg\varphi)$; $\neg\exists x(\varphi) \;\equiv\; \forall x(\neg\varphi)$
3. **Rectify** by renaming bound variables so each occurs only once and no bound variable also occurs free.
4. Pull quantifiers out using the following rules, assuming that x does not occur in α:

   $\alpha \wedge \forall x(\beta) \;\equiv\; \forall x(\alpha \wedge \beta)$; $\alpha \vee \forall x(\beta) \;\equiv\; \forall x(\alpha \vee \beta)$; $\alpha \wedge \exists x(\beta) \;\equiv\; \exists x(\alpha \wedge \beta)$; $\alpha \vee \exists x(\beta) \;\equiv\; \exists x(\alpha \vee \beta)$
5. **Skolemize:** remove existential quantifier $\exists x$ and replace x by $f(u_1, \ldots, u_r)$ where $\forall u_1 \ldots u_r$ are to the left of $\exists x$ and f a new function symbol.

In the EF Game, $\mathcal{G}_m^k(\mathcal{A}, \mathcal{B})$, **Delilah wins** if after every step, $j \leq m$ the function, η_j that maps $c^{\mathcal{A}} \mapsto c^{\mathcal{B}}$ for constant symbols $c \in \Sigma$ and $x_i^{\mathcal{A}} \mapsto x_i^{\mathcal{B}}$ for all pebbles x_i that have been placed so far, is an isomorphism of the induced substructures. **Samson wins** if at some step, η_j is not an isomorphism.

$\mathcal{A} \sim_m^k \mathcal{B}$ means **Delilah has a winning strategy** for $\mathcal{G}_m^k(\mathcal{A}, \mathcal{B})$. ($\mathcal{A} \equiv_m^k \mathcal{B}$) means that $\mathcal{A}$ and $\mathcal{B}$ agree on all formulas in $\mathcal{L}_m^k$, i.e., having at most k variables and quantifier rank $\leq m$.

Fund. Thm of EF Games: Finite relational, Σ, $\mathcal{A}, \mathcal{B} \in$ STRUC[Σ], $(\mathcal{A} \equiv_m^k \mathcal{B}) \Leftrightarrow (\mathcal{A} \sim_m^k \mathcal{B})$

$\mathcal{A}$ and $\mathcal{B}$ are **isomorphic** ($\mathcal{A} \cong \mathcal{B}$) iff exists $\eta : |\mathcal{A}| \xrightarrow[\text{onto}]{1:1} |\mathcal{B}|$ such that

$$\text{forall } P \in \Sigma, e_1, \ldots, e_a \in |\mathcal{A}| \quad \big((e_1, \ldots, e_a) \in P^{\mathcal{A}} \quad \Leftrightarrow \quad (\eta(e_1), \ldots, \eta(e_a)) \in P^{\mathcal{B}} \big) \qquad \&$$

$$\text{forall } f \in \Sigma, e_1, \ldots, e_r \in |\mathcal{A}| \quad \big(\eta(f^{\mathcal{A}}(e_1, \ldots, e_r)) \quad = \quad f^{\mathcal{B}}(\eta(e_1), \ldots, \eta(e_r)) \big)$$