Definitions for 501, especially where I differ from Sipser

**Turing Machines:** A TM \( M = (Q, \delta, q_0) \), where \( \delta : Q \times \Gamma \rightarrow Q_h \times \Gamma \times \{-1, 0, 1\} \).

Here \( Q_h \overset{\text{def}}{=} Q \cup \{h\} \) where \( h \) means halt. We always have \( \Sigma = \{0, 1\} \) and \( \Gamma = \{0, 1, \triangleright, \sqsubseteq\} \), where \( \triangleright \) is the left marker – it always and only occurs as the left-most symbol of the tape. The symbol \( \sqsubseteq \) represents a blank square.

The initial instantaneous description of any TM computation is,

\[
\text{ID}_0 \overset{\text{def}}{=} (q_0, \triangleright), w_1, \ldots, w_n, \sqsubseteq
\]

meaning that the TM is in its start state, \( q_0 \), looking at the left marker, its input is \( w = w_1 \cdots w_n \in \Sigma^n \). The rightmost \( \sqsubseteq \) represents infinitely many empty cells to its right. We always use \( n = |w| \) for the **length of the input**.

For any TM, \( M \), we slightly abuse notation and let \( M \) also refer to the partial function computed by \( M \), i.e.,

\[
M : \text{dom}(M) \rightarrow \Sigma^*,
\]

defined as follows:

\[
M(w) = \begin{cases} 
  y \in \Sigma^* & \text{if } M \text{ on input } \triangleright w \sqsubseteq \text{ eventually halts with output beginning } \triangleright y \sqsubseteq \\
  \uparrow & \text{otherwise}
\end{cases}
\]

**Def.** Let \( f \) be a partial function from \( \Sigma^* \) to \( \Sigma^* \). We say that \( f \) is a **partial, computable function** iff \( \exists \) TM \( M \) s.t. \( (\forall w \in \Sigma^*)(f(w) = M(w)) \). Let \( \text{dom}(M) = \{w \in \Sigma^* \mid M(w) \neq \uparrow\} \). If \( \text{dom}(M) = \Sigma^* \) then \( M \) is **total**, otherwise \( M \) is **strictly partial**.

**Def.** [Partial and total characteristic functions]. For any \( S \subseteq \Sigma^* \), define the total and partial characteristic functions of \( S \) as follows:

\[
\chi_S(x) = \begin{cases} 
  1 & \text{if } x \in S \\
  0 & \text{otherwise}
\end{cases}
\]

\[
p_S(x) = \begin{cases} 
  1 & \text{if } x \in S \\
  \uparrow & \text{otherwise}
\end{cases}
\]

**Def.** [Turing computable sets] Let \( A \subseteq \Sigma^* \). We say that \( A \) is **Turing computable** (synonyms are **computable, decidable, solvable**, and **recursive**), iff \( \chi_A \) is computable, i.e., for some TM, \( M \), \( (\forall w \in \Sigma^*)(M(w) = \chi_A(w)) \).

**Def.** [Turing recognizable sets]. For any TM, \( M \), let

\[
\mathcal{L}(M) = \{ w \in \Sigma^* \mid M(w) = 1 \}.
\]

For any \( A \subseteq \Sigma^* \), we say that \( A \) is **Turing recognizable** (synonyms are **Turing enumerable, recursive enumerable, r.e.**) iff \( \exists \) TM \( M \) s.t. \( A = \mathcal{L}(M) \).

**Proposition:** For all \( A \subseteq \Sigma^* \), \( A \) is r.e. iff \( p_A \) is computable.