Definitions for 501, especially where I differ from Sipser

Turing Machines: A TM $M = (Q, \delta, q_0)$, where $\delta : Q \times \Gamma \to Q_h \times \Gamma \times \{-1, 0, 1\}$.

Here $Q_h \overset{\text{def}}{=} Q \cup \{h\}$ where $h$ means halt. We always have $\Sigma = \{0, 1\}$ and $\Gamma = \{0, 1, \triangleright, \sqcup\}$, where $\triangleright$ is the left marker – it always and only occurs as the left-most symbol of the tape. The symbol “$\sqcup$” represents a blank square.

The initial instantaneous description of any TM computation is,

$\text{ID}_0 \overset{\text{def}}{=} (q_0, \triangleright), w_1, \ldots, w_n, \sqcup$

meaning that the TM is in its start state, $q_0$, looking at the left marker, its input is $w = w_1 \cdots w_n \in \Sigma^n$. The rightmost $\sqcup$ represents infinitely many empty cells to its right. We always use $n = |w|$ for the length of the input.

For any TM, $M$, we slightly abuse notation and let $M$ also refer to the partial function computed by $M$, i.e., $M : \text{dom}(M) \to \Sigma^*$, defined as follows:

$$M(w) = \begin{cases} y \in \Sigma^* & \text{if } M \text{ on input } \triangleright w \sqcup \text{ eventually halts with output beginning } \triangleright y \sqcup \\ \n & \text{otherwise} \end{cases}$$

Def. Let $f$ be a partial function from $\Sigma^*$ to $\Sigma^*$. We say that $f$ is a partial, computable function iff $\exists$ TM $M$ s.t. $(\forall w \in \Sigma^*)(f(w) = M(w))$. Let $\text{dom}(M) = \{w \in \Sigma^* \mid M(w) \neq \n\}$. If $\text{dom}(M) = \Sigma^*$ then $M$ is total, otherwise $M$ is strictly partial.

Def. [Partial and total characteristic functions]. For any $S \subseteq \Sigma^*$, define the total and partial characteristic functions of $S$ as follows:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases} \quad p_S(x) = \begin{cases} 1 & \text{if } x \in S \\ \n & \text{otherwise} \end{cases}$$

Def. [Turing computable sets] Let $A \subseteq \Sigma^*$. We say that $A$ is Turing computable (synonyms are computable, decidable, solvable, and recursive), iff $\chi_A$ is computable, i.e., for some TM, $M$, $(\forall w \in \Sigma^*)(M(w) = \chi_A(w))$.

Def. [Turing recognizable sets]. For any TM, $M$, let

$$\mathcal{L}(M) = \{w \in \Sigma^* \mid M(w) = 1\}.$$

For any $A \subseteq \Sigma^*$, we say that $A$ is Turing recognizable (synonyms are Turing enumerable, recursive enumerable, r.e.) iff $\exists$ TM $M$ s.t. $A = \mathcal{L}(M)$.

Proposition: For all $A \subseteq \Sigma^*$, $A$ is r.e. iff $p_A$ is computable.

Theorem: There exists a Universal TM, $U$, s.t. $\forall n, w \in \mathbb{Z}^+$,

$$U(n, w) = M_n(w), \text{ i.e., } U \text{ on input } (n, w) \text{ does exactly what } M_n \text{ does on input } w.$$

Def. $K = \{n \mid n \in \mathcal{L}(M_n)\} = \{n \mid M_n(n) = 1\}$.
Theorem: $K$ is r.e. but not recursive.

Def. $W_i = \mathcal{L}(M_i)$. $W_i$ is the $i$th r.e. set. A set is r.e. iff it is $W_i$ for some $i$.

Note that $K$ was constructed so that for all $j$, $j \in K \iff j \not\in W_j$.

Thus, by construction, $K \neq W_j$, for every $j$. Thus $K$ is not r.e.

$K = D$, the Diagonalization language that Sipser defines in 4.2.

Def. $\text{HALT} = \{(n, w) \mid M_n(w) \downarrow\}$; $A_{TM} = \{(n, w) \mid M_n(w) = 1\}$

Corollary: $\text{HALT}$ and $A_{TM}$ are r.e. but not recursive.

Fundamental Theorem of r.e. Sets Let $A \subseteq \Sigma^*$. T.F.A.E.

1. $\exists$ TM $M_1$, $A = \text{dom}(M_1) = \{i \in \mathbb{N} \mid M_1(i) \downarrow\}$.
2. $\exists$ TM $M_2$, $A = \text{range}(M_2) = \{M_2(i) \mid i \in \mathbb{N}\}$.
3. $\exists$ TM $M_3$, $A = \emptyset$ or $A = \text{range}(M_3)$ and $M_3$ is total.
4. $\exists$ TM $M_4$, $A = \mathcal{L}(M_4)$.