CS501: Formal Language Theory

L20: Computational Complexity
**Def:** A set $A \subseteq \Sigma^*$ is in $\text{DTIME}[t(n)]$ iff there exists a deterministic, multi-tape TM, $M$, and a constant $c$, such that,

1. $A = L(M)$, and
2. $\forall w \in \Sigma^*$, $M(w)$ halts within $c \cdot t(|w|)$ steps.

**Def:** A set $A \subseteq \Sigma^*$ is in $\text{DSPACE}[s(n)]$ iff there exists a deterministic, multi-tape TM, $M$, and a constant $c$, such that,

1. $A = L(M)$, and
2. $\forall w \in \Sigma^*$, $M(w)$ uses at most $c \cdot s(|w|)$ work-tape cells.

(The input tape is considered “read-only” and not counted as space used.)
**Thm:** For any functions $t(n) \geq n, \ s(n) \geq \log n,$

\[
\text{DTIME}[t(n)] \subseteq \text{DSPACE}[t(n)] \\
\text{DSPACE}[s(n)] \subseteq \text{DTIME}[2^{O(s(n))}]
\]

**Proof.**

Let $M$ be a $\text{DSPACE}(n)]$ TM, $w \in \Sigma^*_0, \ n = |w|$

$M(w)$ has $k$ tapes and uses at most $cs(n)$ work-tape cells.

$M(w)$ has at most $2^{k's(n)}$ possible configurations:

\[
|Q| \cdot (n + cs(n) + 2)^k \cdot |\Sigma|^{cs(n)} < 2^{k's(n)}
\]

# states \cdot # head positions \cdot # tape contents

Thus, after $2^{k's(n)}$ steps, $M(w)$ must be in an infinite loop. □
$$\text{NTIME}[t(n)] \equiv \text{problems accepted by NTMs in time } t(n)$$

$$\text{NP} \equiv \text{NTIME}[n^{O(1)}] \equiv \bigcup_{i=1}^{\infty} \text{NTIME}[n^i]$$

**Theorem**  For any function $t(n)$,

$$\text{DTIME}[t(n)] \subseteq \text{NTIME}[t(n)] \subseteq \text{DSPACE}[t(n)] \subseteq \text{DTIME}[2^{O(t(n))}]$$

**Cor:** $L \subseteq P \subseteq NP \subseteq \text{PSPACE}$
For any complexity class $C$, define $F(C)$, the total, polynomially-bounded functions computable in $C$ as follows:

$$F(C) = \left\{ h : \Sigma^* \rightarrow \Sigma^* \mid \exists k \forall x (|h(x)| \leq k|x|^k \text{ and bit-graph}(h) \in C) \right\}$$

bit-graph$(h) = \left\{ \langle x, i, b \rangle \mid \text{bit } i \text{ of } h(x) \text{ is } b \right\}$

Idea: $f \in F(C)$ iff

1. $f$ is polynomially bounded, and,
2. bit $i$ of $f(w)$ is uniformly computable in $C$ and co-$C$. 

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