CS501: Formal Language Theory

L30: P and PSPACE
Def: In an alternating graph, $G = (V, A, E)$, $i$ reaches $t$ iff

1. $i = t$, or
2. $i$ is an $\exists$ node and for some edge, $\langle i, a \rangle \in E$, $a$ is reaches $t$, or,
3. $i$ is a $\forall$ node and there is an edge leaving $i$ and for all edges, $\langle i, a \rangle \in E$, $a$ is reaches $t$. 
\[ \text{AREACH} = \{ G = (V, A, E, s, t) \mid s \text{ reaches } t \} \]

**Prop:** AREACH is \( P \) complete.

**Proof:** A arbitrary \( \text{ASPACE}[\log n] \) TM; \( w \) input; \( n = |w| \),

\( (w \in L(A)) \Leftrightarrow (\text{CompGraph}(A, w) \in \text{AREACH}) \)

**Cor:** CVP and MCVP are \( P \) complete.

**Proof:** Easy to see that \( \text{AREACH} \leq \text{MCVP} \leq \text{CVP} \).
PSPACE = DSPACE[$n^{O(1)}$] = NSPACE[$n^{O(1)}$]

- PSPACE consists of what we could compute with a feasible amount of hardware, but with no time limit.

- PSPACE is a large and very robust complexity class.

- With polynomially many bits of memory, we can search any implicitly-defined graph of exponential size. Succinct-REACH is PSPACE complete.

- We can search the game tree of any board game whose configurations are polynomial size. This leads to PSPACE complete games.
Recall from Lecture 29: \( \text{PSPACE} = \text{ATIME}[n^{O(1)}] \)

Recall QSAT, the quantified satisfiability problem.

**Prop:** QSAT is PSPACE-complete.

**Proof:** QSAT \( \in \text{ATIME}[n] \subseteq \text{PSPACE} \) (Lecture 28).
Claim: QSAT is hard for $\text{ATIME}[n^k]$. 

Let $M$ be an $\text{ATIME}[n^k]$ TM, $w$ an input, $n = |w|$. 

Let $M$ write down its $n^k$ alternating choices, $c_1 \ c_2 \ldots \ c_{n^k}$. 

Deterministic TM $D$ evaluates the answer, i.e., for all inputs $w$, 

$M(w) = 1 \iff \exists c_1 \ \forall c_2 \cdots \exists c_{n^k} \ (D(\overline{c}, w) = 1)$ 

By Cook’s Theorem $\exists$ reduction $f : L(D) \leq \text{SAT}$: 

$D(\overline{c}, w) = 1 \iff f(\overline{c}, w) \in \text{SAT}$ 

Let the new boolean variables in $f(\overline{c}, w)$ be $d_1 \ldots d_{t(n)}$. 

$M(w) = 1 \iff \exists c_1 \ \forall c_2 \cdots \exists c_{n^e} \ \exists d_1 \cdots \exists d_{t(n)} \ (f(\overline{c}, w)) \in \text{QSAT}$ □
Geography is a two-person game.

1. $E$ chooses a start vertex, $v_1$.
2. $A$ chooses $v_2$, having an edge from $v_1$
3. $E$ chooses $v_3$, have an edge from $v_2$, etc.

No vertex may be chosen twice. Whoever moves last wins.
GEOGRAPHY $\overset{\text{def}}{=} \{ p \in \text{Pos}(\text{Geo}) \mid \exists \text{ has a winning strategy for } p \}$

**Prop:** GEOGRAPHY is PSPACE-complete.

**Proof:** (GEOGRAPHY $\in$ PSPACE): Just search the polynomial-depth game tree. A polynomial-size stack suffices.
Show: $\text{QSAT} \leq \text{GEOGRAPHY}$

Given formula, $\varphi$, build graph $G_{\varphi}$ s.t. $\exists$ chooses existential variables; $\forall$ chooses universal variables.

$$
\varphi \equiv \exists a \forall b \exists c \\
[(a \lor b) \land (\overline{b} \lor c) \land (b \lor \overline{c})]
$$
Succinct representation of graph \( G(n, C, s, t) = (V, E, s, t) \)

\( C \) is a boolean circuit with \( 2n \) inputs and

\[
V = \{ w \mid w \in \{0, 1\}^n \}
\]

\[
E = \{ (w, w') \mid C(w, w') = 1 \}
\]

\( V \) has \( 2^n \) vertices;

Circuit or Algorithm \( C \) gives rule for when \( (w, w') \in E \)

\[
\text{SUCCINCT REACH} = \{ (n, C, s, t) \mid G(n, C, s, t) \in \text{REACH} \}
\]
**Prop:** SUCCINCT REACH ∈ PSPACE

**Proof:** Remember Savitch’s Thm:

\[
\text{REACH} \in \text{NSPACE}[\log n] \subseteq \text{DSPACE}[(\log n)^2]
\]

\[
\text{SUCCINCT REACH} \in \text{NSPACE}[n] \subseteq \text{DSPACE}[n^2] \subseteq \text{PSPACE} \quad \square
\]
Prop: SUCCINCT REACH is PSPACE-complete.

Proof: Let $M$ be a DSPACE[$n^k$] TM, input $w$, $n = |w|$

$$(M(w) = 1) \iff (\text{CompGraph}(M, w) \in \text{REACH})$$

$$\text{CompGraph}(n, w) = (V, E, s, t)$$

$$V = \{ \text{ID} = \langle q, h, p \rangle \mid q \in \text{States}(N), h \leq n, |p| \leq c n^k \}$$

$$E = \{ (\text{ID}_1, \text{ID}_2) \mid \text{ID}_1(w) \xrightarrow{M} \text{ID}_2(w) \}$$

$s = \text{initial ID}$

$t = \text{accepting ID}$

□
Succinct Representation of $\text{CompGraph}(n, w)$:

\[ V = \{ \text{ID} = \langle q, h, p \rangle \mid q \in \text{States}(N), h \leq n, |p| \leq cn^k \} \]

\[ E = \{ (\text{ID}_1, \text{ID}_2) \mid \text{ID}_1(w) \xrightarrow{M} \text{ID}_2(w) \} \]

Let $V = \{0, 1\}^{c'n^k}$

Build circuit $C_w$: on input $u, v \in V$, accept iff $u \xrightarrow{M} v$.

$M(w) = 1 \iff G(c'n^k, C_w, s, t) \in \text{SUCCINCT REACH}$