L27: Savitch’s Theorem
Recall from Last Time: **Thm:** REACH is complete for NL.

A ∈ NL : arbitrary: A = L(N)

∀w ∈ {0, 1}*

(w ∈ L(N)) ⇔ (CompGraph(N, w) ∈ REACH)

\[ f(w) = \text{CompGraph}(N, w) \stackrel{\text{def}}{=} (V, E, s, t) \]

\[ V = \{ \text{ID} = \langle q, h, p \rangle \mid q \in \text{States}(N), h \leq n, |p| \leq c \lceil \log n \rceil \} \]

\[ E = \{ (\text{ID}_1, \text{ID}_2) \mid \text{ID}_1(w) \xrightarrow{N} \text{ID}_2(w) \} \]

s = initial ID

t = accepting ID
Cor:  \( \text{NL} \subseteq \text{P} \)

Proof: \( \text{REACH} \in \text{P} \)

\( \text{P} \) is closed under (logspace) reductions.

i.e., \( (B \in \text{P} \land A \leq B) \implies A \in \text{P} \) \( \square \)
Prop:

\[ \text{NSPACE}[s(n)] \subseteq \text{NTIME}[2^{O(s(n))}] \subseteq \text{DSPACE}[2^{O(s(n))}] \]

We can do much better!
Thm:  \( \text{REACH} \in \text{DSPACE}[(\log n)^2] \)

Proof.

\[ G \in \text{REACH} \iff G \models \text{PATH}(s,t,n) \]
\[ \text{PATH}(x,y,1) \equiv x = y \lor E(x,y) \]
\[ \text{PATH}(x,y,2d) \equiv \exists z \ (\text{PATH}(x,z,d) \land \text{PATH}(z,y,d)) \]

\( S_n(d) = \text{space to check paths of dist. } d \text{ in } n\text{-node graphs} \)

\[ S_n(n) = \log n + S_n(n/2) = O((\log n)^2) \]
Savitch’s Thm: \( \text{DSPACE}[s(n)] \subseteq \text{NSPACE}[s(n)] \subseteq \text{DSPACE}[(s(n))^2] \)

**Proof:** Let \( A \in \text{NSPACE}[s(n)]; \quad A = \mathcal{L}(N) \)

\[ w \in A \iff \text{CompGraph}(N, w) \in \text{REACH} \]

\[ |w| = n; \quad |\text{CompGraph}(N, w)| = 2^{O(s(n))} \]

Testing if \( \text{CompGraph}(N, w) \in \text{REACH} \) takes space,

\[ (\log(|\text{CompGraph}(N, w)|))^2 = (\log(2^{O(s(n))}))^2 \]

\[ = O((s(n))^2) \]

From \( w \) build \( \text{CompGraph}(N, w) \) in \( \text{DSPACE}[s(n)] \). \( \square \)
Thm: \( \text{REACH} \in \text{NL} \)

**Proof:** Fix \( G \), let 
\[ N_d = \left| \left\{ v \mid \text{distance}(s, v) \leq d \right\} \right| \]

**Claim:** The following problems are in \( \text{NL} \):
1. \( \text{dist}(x, d) \): distance \((s, x) \leq d\)
2. \( \text{NDIST}(x, d; m) \): if \( m = N_d \) then \( \neg \text{dist}(x, d) \)

**Proof.**

1. Guess the path of length \( \leq d \) from \( s \) to \( x \).
2. Guess \( m \) vertices, \( v \neq x \), with \( \text{dist}(v, d) \).

\[
c := 0;
\text{for } v := 1 \text{ to } n \text{ do } \{ \quad // \text{nondeterministically}
\quad (\text{dist}(v, d) \&\& v \neq x; c++) \quad ||
\quad (\text{no-op})
\}\]
\[
\text{if } (c == m) \text{ then ACCEPT}
\]
Claim: We can compute $N_d$ in NL.

Proof: By induction on $d$.

Base case: $N_0 = 1$

Inductive step: Suppose we have $N_d$.

1. $c := 0$;
2. for $v := 1$ to $n$ do { // nondeterministically
   3. ( dist($v, d + 1$); $c++$ ) ||
   4. ( $\forall z (\text{NDIST}(z, d; N_d) \lor (z \neq v \land \neg E(z, v)))$)
   5. }
3. $N_{d+1} := c$

$G \in \text{REACH} \iff \text{NDIST}(t, n; N_n)$
Immerman-Szelepcsényi Thm:

If \( s(n) \geq \log n \), Then, \( \text{NSPACE}[s(n)] = \text{co-NSPACE}[s(n)] \)

Proof.

Let \( A \in \text{NSPACE}[s(n)] \); \( A = \mathcal{L}(\mathcal{N}) \)

\( w \in A \iff \text{CompGraph}(\mathcal{N}, w) \in \text{REACH} \)

\( |w| = n; \quad |\text{CompGraph}(\mathcal{N}, w)| = 2^{O(s(n))} \)

Testing if \( \text{CompGraph}(\mathcal{N}, w) \in \text{REACH} \) takes space,

\[
\log(|\text{CompGraph}(\mathcal{N}, w)|) = \log(2^{O(s(n))}) = O(s(n))
\]