L25: Hierarchy Theorems
Hierarchy Theorems

**If** \( f(n) \) is a \( C \)-constructible function;

\( C \) is \( \text{DSPACE}, \text{NSPACE}, \text{DTIME}, \) or \( \text{NTIME} \); and,

if \( g(n) \) is sufficiently smaller than \( f(n) \)

**Then** \( C[g(n)] \subsetneq C[f(n)] \).

**\( g(n) \) sufficiently smaller**

than \( f(n) \) means:

\[
\lim_{n \to \infty} \left( \frac{g(n)}{f(n)} \right) = 0
\]

\[
\lim_{n \to \infty} \left( \frac{g(n) \log(g(n))}{f(n)} \right) = 0
\]

\( C = \text{DSPACE}, \text{NSPACE}, \text{NTIME} \)

\( C = \text{DTIME} \)

**Corollaries:**

\( \text{DSPACE}[n] \subsetneq \text{DSPACE}[n \log \log n] \)

\( \text{DTIME}[n] \subsetneq \text{DTIME}[n \log n]^{1.01} \)
Definition:

Function \( f : \mathbb{N} \rightarrow \mathbb{N} \) is **C-constructible** if the map

\[
1^n \mapsto f(n)
\]

is computable in the complexity class \( \mathbb{C}[f(n)] \).

For example a function \( f(n) \) is **DSPACE-constructible** if the function \( f(n) \) can be deterministically computed from the input \( 1^n \), using space at most \( O[f(n)] \).

**Fact:** All reasonable functions greater than or equal to \( \log n \) are **DSPACE-constructible**, and all reasonable functions greater than or equal to \( n \) are **DTIME-constructible**.
Space Hierarchy Thm: If $f \geq \log n$ is space constructible and

$$\lim_{n \to \infty} \left( \frac{g(n)}{f(n)} \right) = 0,$$

Then $\text{DSPACE}[g(n)] \subsetneq \text{DSPACE}[f(n)]$.

Proof: Build $\text{DSPACE}[f(n)]$ machine, $D$, on input: $w$, $n = |w|$

1. Mark off $6f(n)$ tape cells, ($f$ space constructible)
2. Simulate $M_w(w)$ using space $3f(n)$, time $\leq 2^{3f(n)}$
3. if ($M_w(w)$ needs more space or time): return (17)
4. else if ($M_w(w) = \text{accept}$): reject
5. else accept  // ($M_w(w) = \text{reject}$)

<table>
<thead>
<tr>
<th>space to simulate $M_w(w)$</th>
<th>counter</th>
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<tbody>
<tr>
<td>$3f(n)$</td>
<td>$3f(n)$</td>
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Claim: \( L(D) \in \text{DSPACE}[f(n)] - \text{DSPACE}[g(n)] \)

**Proof:** \( L(D) \in \text{DSPACE}[f(n)] \) by construction.

**Suppose** \( L(D) \in \text{DSPACE}[g(n)] \).

Let \( L(M_w) = L(D) \), \( M_w \) uses \( cg(n) \) space.

Choose \( N \) s.t. \( \forall n > N \left( cg(n) < f(n) \right) \).

Choose \( w' \), \( M_{w'}(\cdot) = M_w(\cdot) \), \( |w'| > N \)

On input \( w' \), \( D \) successfully simulates \( M_{w'}(w') \) in \( 3f(n) \) space and \( 2^{3f(n)} \) time.

\[
\begin{align*}
w' \in L(D) &\iff w' \not\in L(M_{w'}) \iff w' \not\in L(M_w) \iff w' \not\in L(D) \\
&\iff \square
\end{align*}
\]
**Hierarchy Theorems**

If $f(n)$ is a $\mathbf{C}$-constructible function;

$\mathbf{C}$ is $\text{DSPACE}$, $\text{NSPACE}$, $\text{DTIME}$, or $\text{NTIME}$; and,

if $g(n)$ is sufficiently smaller than $f(n)$

Then $\mathbf{C}[g(n)] \subsetneq \mathbf{C}[f(n)]$.

$g(n)$ **sufficiently smaller** than $f(n)$ means:

$$\lim_{n \to \infty} \left( \frac{g(n)}{f(n)} \right) = 0$$

$$\lim_{n \to \infty} \left( \frac{g(n) \log(g(n))}{f(n)} \right) = 0$$

$\mathbf{C} = \text{DSPACE}$, $\text{NSPACE}$, $\text{NTIME}$

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**Corollaries:**

$\text{DSPACE}[n] \subsetneq \text{DSPACE}[n \log \log n]$

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