CS250: Discrete Math for Computer Science

L6: CNF and Natural Deduction for PropCalc

How to Simplify a PropCalc Formula:

$$(p \to q) \to ((q \to r) \wedge p) \quad \equiv$$

How to Simplify a PropCalc Formula:

1. Get rid of →'s using **def. of implication**.

$$\begin{aligned}(p \to q) \to ((q \to r) \wedge p) &\equiv (\sim p \vee q) \to ((\sim q \vee r) \wedge p)\\ &\equiv \sim(\sim p \vee q) \vee ((\sim q \vee r) \wedge p)\end{aligned}$$

How to Simplify a PropCalc Formula:

1. Get rid of →'s using **def. of implication**.
2. Push ∼'s all the way inside using **de Morgan**.
   Now its in Negation Normal Form (**NNF**).

$$\begin{array}{lcl} (p \to q) \to ((q \to r) \wedge p) & \equiv & (\sim p \vee q) \to ((\sim q \vee r) \wedge p) \\ & \equiv & \sim(\sim p \vee q) \vee ((\sim q \vee r) \wedge p) \\ \textbf{NNF} & \equiv & (p \wedge \sim q) \vee ((\sim q \vee r) \wedge p) \end{array}$$

How to Simplify a PropCalc Formula:

1. Get rid of $\to$'s using **def. of implication**.
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   Now its in Negation Normal Form (**NNF**).
3. Use distributive, associative, commutative to put in either
   Disjunctive Normal Form (**DNF**: $\vee$'s of $\wedge$'s of literals)

   A **literal** is a prop variable or its negation, e.g., p, $\sim q$.

$$\begin{array}{lcl}
(p \to q) \to ((q \to r) \wedge p) & \equiv & (\sim p \vee q) \to ((\sim q \vee r) \wedge p) \\
 & \equiv & \sim(\sim p \vee q) \vee ((\sim q \vee r) \wedge p) \\
\textbf{NNF} & \equiv & (p \wedge \sim q) \vee ((\sim q \vee r) \wedge p) \\
\textbf{NNF} & \equiv & (p \wedge \sim q) \vee ((\sim q \wedge p) \vee (r \wedge p)) \\
\textbf{DNF} & \equiv & (p \wedge \sim q) \vee (p \wedge r)
\end{array}$$

How to Simplify a PropCalc Formula:

1. Get rid of $\rightarrow$'s using **def. of implication**.
2. Push $\sim$'s all the way inside using **de Morgan**.
   Now its in Negation Normal Form (**NNF**).
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   Disjunctive Normal Form (**DNF**: $\vee$'s of $\wedge$'s of literals)
   Conjunctive Normal Form (**CNF**: $\wedge$'s of $\vee$'s of literals)
   A **literal** is a prop variable or its negation, e.g., p, $\sim q$.

$$(p \rightarrow q) \rightarrow ((q \rightarrow r) \wedge p) \quad \equiv \quad (\sim p \vee q) \rightarrow ((\sim q \vee r) \wedge p)$$

$$\equiv \quad \sim(\sim p \vee q) \vee ((\sim q \vee r) \wedge p)$$

NNF $\quad \equiv \quad (p \wedge \sim q) \vee ((\sim q \vee r) \wedge p)$

NNF $\quad \equiv \quad (p \wedge \sim q) \vee ((\sim q \wedge p) \vee (r \wedge p))$

DNF $\quad \equiv \quad (p \wedge \sim q) \vee (p \wedge r)$

CNF $\quad \equiv \quad p \wedge (\sim q \vee r)$

iClicker 6.1 Which of the following formulas is in CNF (conjunction of disjunctions of literals)?

A: $(p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge s) \vee (q \wedge r \wedge \sim s)$

B: $(\sim p \vee q \vee \sim r) \wedge (p \vee \sim q \vee \sim s) \wedge (\sim q \vee \sim r \vee s)$

C: $(p \wedge \sim q) \vee ((\sim q \vee p) \wedge (r \vee p))$

D: $\sim(\sim p \vee q) \vee ((\sim q \vee r) \wedge p)$

Natural Deduction

R6: Our PropCalc proof rules are slightly different from Epp's proof rules.

	introduction	elimination
$\wedge$	$\dfrac{p \quad q}{p \wedge q}$	$\dfrac{p \wedge q}{p}$ $\quad$ $\dfrac{p \wedge q}{q}$
$\vee$	$\dfrac{p}{p \vee q}$ $\quad$ $\dfrac{q}{p \vee q}$	$\dfrac{p \vee q \quad p \vdash r \quad q \vdash r}{r}$
$\rightarrow$	$\dfrac{p \vdash q}{p \rightarrow q}$	$\dfrac{p \rightarrow q \quad p}{q}$ $\quad$ $\dfrac{p \rightarrow q \quad \sim q}{\sim p}$
$\mathbf{F}$	$\dfrac{p \quad \sim p}{\mathbf{F}}$	$\dfrac{p \vdash \mathbf{F}}{\sim p}$ $\quad$ $\dfrac{\sim p \vdash \mathbf{F}}{p}$
$\sim\sim$	$\dfrac{p}{\sim\sim p}$	$\dfrac{\sim\sim p}{p}$

Natural Deduction rule: $\rightarrow$-introduction

1	p	
2		
3	q	
4	$p \rightarrow q$	$\rightarrow$-i, 1–3

$$\frac{p \vdash q}{p \rightarrow q}$$

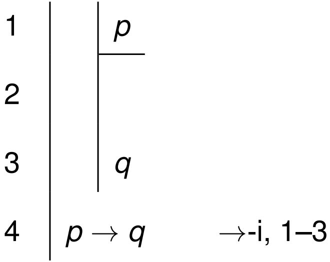
Natural Deduction rule: →-introduction

$$
\begin{array}{r|ll}
1 & \quad | \; \underline{p} & \\
2 & \quad | & \\
3 & \quad | \; q & \\
4 & p \to q & \quad \to\text{-i, } 1\text{–}3
\end{array}
$$

$$\frac{p \vdash q}{p \to q}$$

Notation: $p \vdash q$ (p **proves** q): From assumption p, can prove q.

Natural Deduction rule: $\to$-introduction



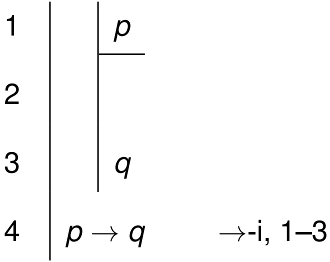
1	p	
2		
3	q	
4	$p \to q$	$\to$-i, 1–3

$$\frac{p \vdash q}{p \to q}$$

Notation: $p \vdash q$ (p **proves** q): From assumption p, can prove q.

Proposition: $\to$-i is sound, i.e., if from assumption p we can prove q, then every world satisfies $p \to q$.

Natural Deduction rule: →-introduction



1 p

2

3 q

4 $p \to q$ →-i, 1–3

$$\frac{p \vdash q}{p \to q}$$

Notation: $p \vdash q$ (p **proves** q): From assumption p, can prove q.

Proposition: →-i is sound, i.e., if from assumption p we can prove q, then every world satisfies $p \to q$.

Proof.

Since $p \vdash q$, and by the soundness of the proof rules so far, we know that every world that satisfies p must also satisfy q. Thus every world satisfies $p \to q$. □

Natural Deduction rule: →-introduction

1	p	
2		
3	q	
4	$p \to q$	→-i, 1–3

$$\frac{p \vdash q}{p \to q}$$

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More about this once we have studied inductive proofs.

Example use of $\to$-introduction

1	p	
2	$r \lor p$	$\lor$-i, 1
3	$p \to (r \lor p)$	$\to$-i, 1–2

Example use of $\rightarrow$-introduction

1	p	
2	$r \lor p$	$\lor$-i, 1
3	$p \rightarrow (r \lor p)$	$\rightarrow$-i, 1–2

Thus, $\vdash p \rightarrow (r \lor p)$

Natural Deduction rule: **F**-e   Proof by Contradiction

1		$\sim p$	
2			
3		$\mathbf{F}$	
4	p		**F**-e, 1–3

$$\frac{\sim p \vdash \mathbf{F}}{p} \qquad \frac{p \vdash \mathbf{F}}{\sim p}$$

Natural Deduction rule: **F**-e   Proof by Contradiction

1		$\sim p$	
2			
3		**F**	
4	p		**F**-e, 1–3

$$\frac{\sim p \vdash \mathbf{F}}{p} \qquad \frac{p \vdash \mathbf{F}}{\sim p}$$

Proposition: F-e is sound, i.e., if $\sim p \vdash$ F then every world satisfies p.

Natural Deduction rule: **F**-e   Proof by Contradiction

1	$\quad\mid \sim p$	
2	$\quad\mid$	
3	$\quad\mid \mathbf{F}$	
4	p	**F**-e, 1–3

$$\frac{\sim p \vdash \mathbf{F}}{p} \qquad \frac{p \vdash \mathbf{F}}{\sim p}$$

Proposition: F-e is sound, i.e., if $\sim p \vdash \mathbf{F}$ then every world satisfies p.

Proof.

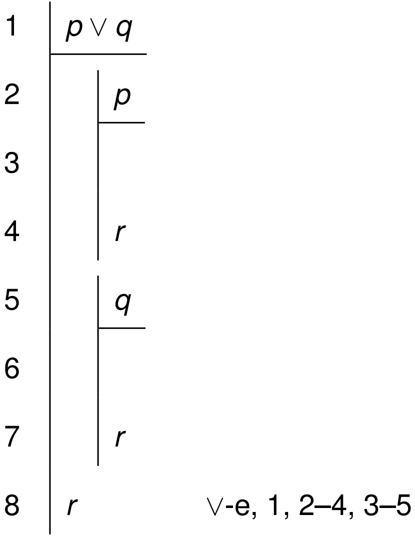
Since $\sim p \vdash \mathbf{F}$, by the soundness of the proof rules so far, we know that every world that satisfies $\sim p$ must also satisfy F. But no world satisfies F. Thus every world satisfies p. □

Natural Deduction rule: $\vee$-e

1	$p \vee q$	
2	$\quad p$	
3		
4	$\quad r$	
5	$\quad q$	
6		
7	$\quad r$	
8	r	$\vee$-e, 1, 2–4, 3–5

$$\frac{p \vee q \quad p \vdash r \quad q \vdash r}{r}$$

Natural Deduction rule: $\vee$-e



1 $p \vee q$

2 p

3

4 r

5 q

6

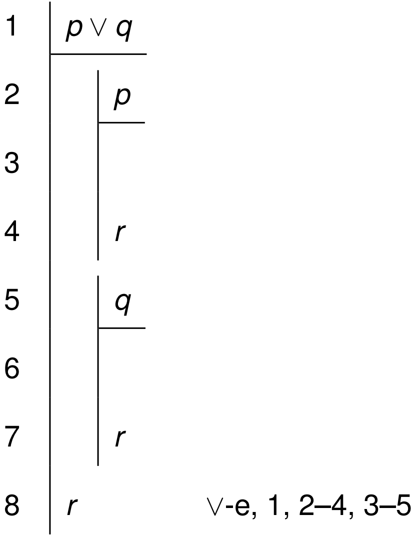
7 r

8 r $\vee$-e, 1, 2–4, 3–5

$$\frac{p \vee q \quad p \vdash r \quad q \vdash r}{r}$$

Proposition: $\vee$-e is sound.

Natural Deduction rule: $\lor$-e



1	$p \lor q$	
2	p	
3		
4	r	
5	q	
6		
7	r	
8	r	$\lor$-e, 1, 2–4, 3–5

$$\frac{p \lor q \quad p \vdash r \quad q \vdash r}{r}$$

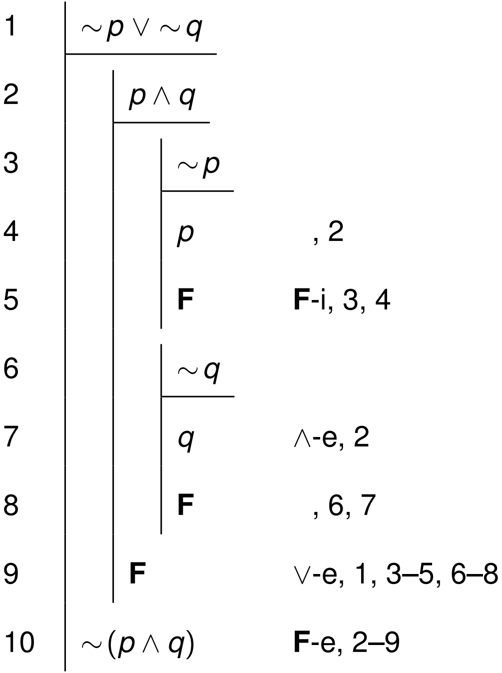
Proposition: $\lor$-e is sound.

Proof.

Since $p \vdash r$ and $q \vdash r$, every world that satisfies p or satisfies q satisfies r. Thus every world that satisfes $p \lor q$ satisfies r.

□

1	$\sim p \vee \sim q$			
2		$p \wedge q$		
3			$\sim p$	
4			p	, 2
5			**F**	**F**-i, 3, 4
6			$\sim q$	
7			q	$\wedge$-e, 2
8			**F**	, 6, 7
9		**F**		$\vee$-e, 1, 3–5, 6–8
10	$\sim(p \wedge q)$			**F**-e, 2–9



1	$\sim p \vee \sim q$	
2	$p \wedge q$	
3	$\sim p$	
4	p	, 2
5	**F**	**F**-i, 3, 4
6	$\sim q$	
7	q	$\wedge$-e, 2
8	**F**	, 6, 7
9	**F**	$\vee$-e, 1, 3–5, 6–8
10	$\sim(p \wedge q)$	**F**-e, 2–9

iClicker 6.2 What is the justification for line 4 ?

A: $\wedge$**-i**
B: $\wedge$**-e**
C: $\vee$**-i**
D: $\vee$**-e**

1	$\mid \sim p \lor \sim q$	
2	$\mid \mid p \land q$	
3	$\mid \mid \mid \sim p$	
4	$\mid \mid \mid p$	, 2
5	$\mid \mid \mid \mathbf{F}$	**F**-i, 3, 4
6	$\mid \mid \mid \sim q$	
7	$\mid \mid \mid q$	$\land$-e, 2
8	$\mid \mid \mid \mathbf{F}$	, 6, 7
9	$\mid \mid \mathbf{F}$	$\lor$-e, 1, 3–5, 6–8
10	$\mid \sim(p \land q)$	**F**-e, 2–9

iClicker 6.2 What is the justification for line 4 ?

- **A:** **$\land$-i**
- **B:** **$\land$-e**
- **C:** **$\lor$-i**
- **D:** **$\lor$-e**

iClicker 6.3 What is the justification for line 8 ?

- **A:** **$\land$-i**
- **B:** **$\land$-e**
- **C:** **F-i**
- **D:** **F-e**

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$\vee$	$\dfrac{p}{p \vee q}$ $\dfrac{q}{p \vee q}$	$\dfrac{p \vee q \quad p \vdash r \quad q \vdash r}{r}$
$\rightarrow$	$\dfrac{p \vdash q}{p \rightarrow q}$	$\dfrac{p \rightarrow q \quad p}{q}$ $\dfrac{p \rightarrow q \quad \sim q}{\sim p}$
$\mathbf{F}$	$\dfrac{p \quad \sim p}{\mathbf{F}}$	$\dfrac{p \vdash \mathbf{F}}{\sim p}$ $\dfrac{\sim p \vdash \mathbf{F}}{p}$
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Propositional Equivalence

PropCalc formulas p and q are **equivalent** ($p \equiv q$) iff they agree on every row of their truth tables.

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double negation	$p \equiv \sim\sim p$

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de Morgan	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
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de Morgan	$\sim(p \wedge q)$	$\equiv$	$\sim p \vee \sim q$
def. of implication	$p \rightarrow q$	$\equiv$	$\sim p \vee q$

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def. of implication	$p \rightarrow q$	$\equiv$	$\sim p \vee q$
contrapositive	$p \rightarrow q$	$\equiv$	$\sim q \rightarrow \sim p$

Propositional Equivalence

PropCalc formulas p and q are **equivalent** ($p \equiv q$) iff they agree on every row of their truth tables.

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Some Important Equivalences (worth memorizing):

double negation	p	$\equiv$	$\sim\sim p$
de Morgan	$\sim(p \vee q)$	$\equiv$	$\sim p \wedge \sim q$
de Morgan	$\sim(p \wedge q)$	$\equiv$	$\sim p \vee \sim q$
def. of implication	$p \rightarrow q$	$\equiv$	$\sim p \vee q$
contrapositive	$p \rightarrow q$	$\equiv$	$\sim q \rightarrow \sim p$
def. of iff	$p \leftrightarrow q$	$\equiv$	$(p \rightarrow q) \wedge (q \rightarrow p)$

More Important Equivalences (worth memorizing):

commutative $p \lor q \equiv q \lor p$

commutative $p \land q \equiv q \land p$

More Important Equivalences (worth memorizing):

commutative $p \lor q \equiv q \lor p$

commutative $p \land q \equiv q \land p$

associative $p \land (q \land r) \equiv (p \land q) \land r$

associative $p \lor (q \lor r) \equiv (p \lor q) \lor r$

More Important Equivalences (worth memorizing):

commutative $p \lor q \equiv q \lor p$

commutative $p \land q \equiv q \land p$

associative $p \land (q \land r) \equiv (p \land q) \land r$

associative $p \lor (q \lor r) \equiv (p \lor q) \lor r$

distributive $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

distributive $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$

More Important Equivalences (worth memorizing):

commutative	$p \vee q$	$\equiv$	$q \vee p$
commutative	$p \wedge q$	$\equiv$	$q \wedge p$
associative	$p \wedge (q \wedge r)$	$\equiv$	$(p \wedge q) \wedge r$
associative	$p \vee (q \vee r)$	$\equiv$	$(p \vee q) \vee r$
distributive	$p \vee (q \wedge r)$	$\equiv$	$(p \vee q) \wedge (p \vee r)$
distributive	$p \wedge (q \vee r)$	$\equiv$	$(p \wedge q) \vee (p \wedge r)$
excluded middle	$p \vee \sim p$	$\equiv$	$\mathbf{T}$

SAT is an important class.

If formula a has n PVARs, $p_1, \ldots, p_n$, it would **seem to require** about 2^n time in the worst case to test if $a \in$ **SAT**.

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This sort of search problem: exponentially many possibilities, each one easy to verify, corresponds to Nondeterministic Polynomial Time (NP).

In fact, **SAT** is a hardest such problem: NP complete.

You will learn much more about this in CMPSCI 311.

Knights and Knaves [Smullyan, *What Is the Name of This Book?*]

Knights always truthful; Knaves always lie; $A, B \in \{\text{Kt},\text{Kv}\}$

B : "$A \& B$ opposite types"

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$S_1 \stackrel{\text{def}}{=} A$: "B is Kt" $\quad S_2 \stackrel{\text{def}}{=} B$: "$A \& B$ opposite types"

$T_1 \stackrel{\text{def}}{=} B$ is a Kt $\quad T_2 \stackrel{\text{def}}{=} A \& B$ opposite types

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$$\begin{array}{cccccc}
S_1 & \overset{\text{def}}{=} & A : \text{“}B \text{ is Kt”} & S_2 & \overset{\text{def}}{=} & B : \text{“}A\&B \text{ opposite types”} \\
T_1 & \overset{\text{def}}{=} & B \text{ is a Kt} & T_2 & \overset{\text{def}}{=} & A\&B \text{ opposite types} \\
S_1 & = & T_1 \leftrightarrow A \text{ is Kt} & S_2 & = & T_2 \leftrightarrow B \text{ is Kt}
\end{array}$$

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$$S_1 = T_1 \leftrightarrow A \text{ is Kt} \qquad S_2 = T_2 \leftrightarrow B \text{ is Kt}$$

W	A is Kt	B is Kt	T_1	T_2	$T_1 \leftrightarrow A$ is Kt	$T_2 \leftrightarrow B$ is Kt
W_3	1	1	1	0	1	0
W_2	1	0	0	1	0	0
W_1	0	1	1	1	0	1
W_0	0	0	0	0	1	1

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$$
\begin{array}{lclclcl}
S_1 & \overset{\text{def}}{=} & A : \text{“}B \text{ is Kt”} & S_2 & \overset{\text{def}}{=} & B : \text{“}A \& B \text{ opposite types”} \\
T_1 & \overset{\text{def}}{=} & B \text{ is a Kt} & T_2 & \overset{\text{def}}{=} & A \& B \text{ opposite types} \\
S_1 & = & T_1 \leftrightarrow A \text{ is Kt} & S_2 & = & T_2 \leftrightarrow B \text{ is Kt}
\end{array}
$$

W	A is Kt	B is Kt	T_1	T_2	$T_1 \leftrightarrow A$ is Kt	$T_2 \leftrightarrow B$ is Kt
W_3	1	1	1	0	1	0
W_2	1	0	0	1	0	0
W_1	0	1	1	1	0	1
W_0	0	0	0	0	1	1

W_0 is only world satisfying $S_1 \wedge S_2$.

Knights and Knaves [Smullyan, *What Is the Name of This Book?*]

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$$T_1 \overset{\text{def}}{=} B \text{ is a Kt} \qquad T_2 \overset{\text{def}}{=} A\&B \text{ opposite types}$$

$$S_1 = T_1 \leftrightarrow A \text{ is Kt} \qquad S_2 = T_2 \leftrightarrow B \text{ is Kt}$$

W	A is Kt	B is Kt	T_1	T_2	$T_1 \leftrightarrow A$ is Kt	$T_2 \leftrightarrow B$ is Kt
W_3	1	1	1	0	1	0
W_2	1	0	0	1	0	0
W_1	0	1	1	1	0	1
W_0	0	0	0	0	1	1

W_0 is only world satisfying $S_1 \wedge S_2$.

Thus *A* and *B* are both Knaves.

R6 Quiz Answers: Match the Epp Proof Rules to the equivalent Natural Deduction Rules.

1. Modus Ponens: $\rightarrow$-e
2. Modus Tollens: $\rightarrow$-e
3. Generalization: $\vee$-i
4. Specialization: $\wedge$-e
5. Conjunction: $\wedge$-i

6, 7. In the following proof, identify the natural deduction rules used in lines 2 and 3.

1	$p \wedge q$	
2	q	$\wedge$-e, 1
3	$(\sim r \ \vee q)$	$\vee$-i, 2

8. Is the following a sound proof rule ?

$$\frac{p \to q \quad q}{p}$$

In answering this, you may consider the worlds shown in this truth table:

W	p	q
W_3	1	1
W_2	1	0
W_1	0	1
W_0	0	0

Not valid: reasoning from the converse fails in world W_1.

9. In Smullyan's Island of Knights and Knaves, two natives C and D approach you but only C speaks. C says: Both of us are knaves. What are C and D? C is a Knave and D is a Knight.

10. In Smullyan's Island of Knights and Knaves, you encounter natives E and F. E says: F is a knave. F says: E is a knave. How many knaves are there? 1