CS250: Discrete Math for Computer Science

L5: PropCalc: Conditional Statements

$p \to q \quad \equiv \quad p$ implies $q \quad \equiv \quad$ if p then $q \quad \equiv \quad \sim p \vee q$

Only the truth values of p and q matter, not the presence or absence of a causal relation between them.

W	p	q	$\sim p$	$\sim q$	$\sim p \vee q$ $p \to q$		$p \leftrightarrow q$ $\sim(p \oplus q)$ $(p \to q) \wedge (q \to p)$
W_3	1	1	0	0	1	1	1
W_2	1	0	0	1	0	1	0
W_1	0	1	1	0	1	0	0
W_0	0	0	1	1	1	1	1

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$p \to q \;\equiv\; \sim q \to \sim p$ **contrapositve** **is equivalent.**

W	p	q	$\sim p$	$\sim q$	$\sim q \to \sim p$ $\sim p \vee q$ $p \to q$		$p \leftrightarrow q$ $\sim(p \oplus q)$ $(p \to q) \wedge (q \to p)$
W_3	1	1	0	0	1	1	1
W_2	1	0	0	1	0	1	0
W_1	0	1	1	0	1	0	0
W_0	0	0	1	1	1	1	1

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$p \to q \ \equiv \ \sim q \to \sim p$ **contrapositve** **is equivalent.**

$p \to q \ \not\equiv \ q \to p$ **converse** **not equivalent.**

W	p	q	$\sim p$	$\sim q$	$\sim q \to \sim p$ $\sim p \vee q$ $p \to q$	$q \to p$	$p \leftrightarrow q$ $\sim(p \oplus q)$ $(p \to q) \wedge (q \to p)$
W_3	1	1	0	0	1	1	1
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W_1	0	1	1	0	1	0	0
W_0	0	0	1	1	1	1	1

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$p \to q \;\equiv\; \sim q \to \sim p$ **contrapositve** **is equivalent.**

$p \to q \;\not\equiv\; q \to p$ **converse** **not equivalent.**

$p \to q \;\not\equiv\; \sim p \to \sim q$ **inverse** **not equivalent.**

W	p	q	$\sim p$	$\sim q$	$\sim q \to \sim p$ $\sim p \vee q$ $p \to q$	$q \to p$ $\sim p \to \sim q$	$p \leftrightarrow q$ $\sim(p \oplus q)$ $(p \to q) \wedge (q \to p)$
W_3	1	1	0	0	1	1	1
W_2	1	0	0	1	0	1	0
W_1	0	1	1	0	1	0	0
W_0	0	0	1	1	1	1	1

$p \rightarrow q \quad \equiv \quad p$ implies $q \quad \equiv \quad$ if p then $q \quad \equiv \quad \sim p \vee q$

Only the truth values of p and q matter, not the presence or absence of a causal relation between them.

$p \rightarrow q \equiv \sim q \rightarrow \sim p$ **contrapositve** **is equivalent.**

$p \rightarrow q \not\equiv q \rightarrow p$ **converse** **not equivalent.**

$p \rightarrow q \not\equiv \sim p \rightarrow \sim q$ **inverse** **not equivalent.**

$q \rightarrow p \equiv \sim p \rightarrow \sim q$ **inverse** is contrapositive of **converse**.

W	p	q	$\sim p$	$\sim q$	$\sim q \rightarrow \sim p$ $\sim p \vee q$ $p \rightarrow q$	$q \rightarrow p$ $\sim p \rightarrow \sim q$	$p \leftrightarrow q$ $\sim (p \oplus q)$ $(p \rightarrow q) \wedge (q \rightarrow p)$
W_3	1	1	0	0	1	1	1
W_2	1	0	0	1	0	1	0
W_1	0	1	1	0	1	0	0
W_0	0	0	1	1	1	1	1

English is ambiguous; PropCalc is precise. Translating between them can be subtle.

		p **implies** q	q **implies** p		
		if p **then** q	**if** q **then** p		p **unless** q
		p **only if** q	p **if** q	p **iff** q	
		p **is sufficient for** q	p **is necessary for** q	p **is necessary and sufficient for** q	
p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$	$\sim q \rightarrow p$
1	1	1	1	1	1
1	0	0	1	0	1
0	1	1	0	0	1
0	0	1	1	1	0

Natural Deduction

R6: Our PropCalc proof rules are slightly different from Epp's proof rules. **Important** for the **R6 quiz**.

	introduction	elimination
$\wedge$	$\dfrac{p \quad q}{p \wedge q}$	$\dfrac{p \wedge q}{p}$ $\quad$ $\dfrac{p \wedge q}{q}$
$\vee$	$\dfrac{p}{p \vee q}$ $\quad$ $\dfrac{q}{p \vee q}$	$\dfrac{p \vee q \quad p \vdash r \quad q \vdash r}{r}$
$\rightarrow$	$\dfrac{p \vdash q}{p \rightarrow q}$	$\dfrac{p \rightarrow q \quad p}{q}$ $\quad$ $\dfrac{p \rightarrow q \quad \sim q}{\sim p}$
F	$\dfrac{p \quad \sim p}{\mathbf{F}}$	$\dfrac{p \vdash \mathbf{F}}{\sim p}$ $\quad$ $\dfrac{\sim p \vdash \mathbf{F}}{p}$
$\sim\sim$	$\dfrac{p}{\sim\sim p}$	$\dfrac{\sim\sim p}{p}$

Natural Deduction rule: $\wedge$-introduction

1	p	
2	q	
3	$p \wedge q$	$\wedge$-i, 1, 2

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Notation: $W \models a$ means that a is true in world, W.

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Notation: $W \models a$ means that a is true in world, W.

Proposition: $\wedge$-i is **sound**, i.e., if $W \models p$ and $W \models q$ then $W \models p \wedge q$.

Natural Deduction rule: $\wedge$-introduction

1	p	
2	q	
3	$p \wedge q$	$\wedge$-i, 1, 2

Notation: $W \models a$ means that a is true in world, W.

Proposition: $\wedge$-i is **sound**, i.e., if $W \models p$ and $W \models q$ then $W \models p \wedge q$.

Proof.

By definition of $\wedge$. $\square$

Natural Deduction rule: $\wedge$-elimination

1	$p \wedge q$	
2	p	$\wedge$-e, 1
3	q	$\wedge$-e, 1

Natural Deduction rule: $\wedge$-elimination

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2	p	$\wedge$-e, 1
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Proposition: $\wedge$-e is **sound**, i.e., if $W \models p \wedge q$ then $W \models p$ and $W \models q$.

Natural Deduction rule: $\wedge$-elimination

1	$p \wedge q$	
2	p	$\wedge$-e, 1
3	q	$\wedge$-e, 1

Proposition: $\wedge$-e is **sound**, i.e., if $W \models p \wedge q$ then $W \models p$ and $W \models q$.

Proof.
By definition of $\wedge$. $\square$

Natural Deduction rule: $\vee$-introduction

1	p	
2	$p \vee q$	$\vee$-i, 1
3	$q \vee p$	$\vee$-i, 1

Natural Deduction rule: $\vee$-introduction

1	p	
2	$p \vee q$	$\vee$-i, 1
3	$q \vee p$	$\vee$-i, 1

Proposition: $\vee$-i is **sound**, i.e., if $W \models p$ then $W \models p \vee q$ and $W \models q \vee p$.

Natural Deduction rule: $\vee$-introduction

1	p	
2	$p \vee q$	$\vee$-i, 1
3	$q \vee p$	$\vee$-i, 1

Proposition: $\vee$-i is **sound**, i.e., if $W \models p$ then $W \models p \vee q$ and $W \models q \vee p$.

Proof.

By definition of $\vee$. $\square$

Natural Deduction rule: $\rightarrow$-elimination

1	$p \rightarrow q$	
2	p	
3	q	$\rightarrow$-e, 1, 2

Natural Deduction rule: $\rightarrow$-elimination

1	$p \rightarrow q$	
2	p	
3	q	$\rightarrow$-e, 1, 2

Proposition: $\rightarrow$-e is **sound**, i.e., if $W \models p \rightarrow q$ and $W \models p$ then $W \models q$.

Natural Deduction rule: $\to$-elimination

1	$p \to q$	
2	p	
3	q	$\to$-e, 1, 2

Proposition: $\to$-e is **sound**, i.e., if $W \models p \to q$ and $W \models p$ then $W \models q$.

Proof.

By definition of $\to$. $\square$

Natural Deduction rule: $\rightarrow$-elimination

1	$p \rightarrow q$	
2	$\sim q$	
3	$\sim p$	$\rightarrow$-e, 1, 2

Natural Deduction rule: $\rightarrow$-elimination

1	$p \rightarrow q$	
2	$\sim q$	
3	$\sim p$	$\rightarrow$-e, 1, 2

Proposition: $\rightarrow$-e is **sound**, i.e., if $W \models p \rightarrow q$ and $W \models \sim q$ then $W \models \sim p$.

Natural Deduction rule: $\to$-elimination

$$
\begin{array}{r|ll}
1 & p \to q & \\
2 & \sim q & \\
\hline
3 & \sim p & \quad\quad \to\text{-e, 1, 2}
\end{array}
$$

Proposition: $\to$-e is **sound**, i.e., if $W \models p \to q$ and $W \models \sim q$ then $W \models \sim p$.

Proof.

If $W \models p \to q$, then $W \models \sim q \to \sim p$, the contrapositive. Then, by definition of $\to$, since $W \models \sim q$, we know that $W \models \sim p$ $\square$

Natural Deduction rule: **F**-introduction

1	p	
2	$\sim p$	
3	**F**	**F**-i, 1, 2

Natural Deduction rule: **F**-introduction

1	p	
2	$\sim p$	
3	**F**	**F**-i, 1, 2

Proposition: **F**-i is **sound**, i.e., if $W \models p$ and $W \models \sim p$ then $W \models \mathbf{F}$.

Natural Deduction rule: **F**-introduction

1	p	
2	$\sim p$	
3	**F**	**F**-i, 1, 2

Proposition: **F**-i is **sound**, i.e., if $W \models p$ and $W \models \sim p$ then $W \models \mathbf{F}$.

Proof.

By definition of $\sim$, if $W \models p$, then $W \not\models \sim p$. Thus, it will never be the case that $W \models p$ and $W \models \sim p$. Thus, this proposition is vacuously true. $\square$

Natural Deduction rule: $\sim\sim$-introduction

1	p	
2	$\sim\sim p$	$\sim\sim$-i, 1

Natural Deduction rule: $\sim\sim$-introduction

1	p	
2	$\sim\sim p$	$\sim\sim$-i, 1

Proposition: $\sim\sim$-i is **sound**, i.e., if $W \models p$ then $W \models \sim\sim p$.

Natural Deduction rule: $\sim\sim$-introduction

1	p	
2	$\sim\sim p$	$\sim\sim$-i, 1

Proposition: $\sim\sim$-i is **sound**, i.e., if $W \models p$ then $W \models \sim\sim p$.

Proof.

$\sim\sim p \equiv p$ $\square$

Natural Deduction rule: $\sim\sim$-elimination

1	$\sim\sim p$	
2	p	$\sim\sim$-e, 1

Natural Deduction rule: $\sim\sim$-elimination

1	$\sim\sim p$	
2	p	$\sim\sim$-e, 1

Proposition: $\sim\sim$-e is **sound**, i.e., if $W \models \sim\sim p$ then $W \models p$.

Natural Deduction rule: $\sim\sim$-elimination

1	$\sim\sim p$	
2	p	$\sim\sim$-e, 1

Proposition: $\sim\sim$-e is **sound**, i.e., if $W \models \sim\sim p$ then $W \models p$.

Proof.

$\sim\sim p \equiv p$ $\square$

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$\rightarrow$	$\dfrac{p \vdash q}{p \rightarrow q}$	$\dfrac{p \rightarrow q \quad p}{q} \quad \dfrac{p \rightarrow q \quad \sim q}{\sim p}$
F	$\dfrac{p \quad \sim p}{\mathbf{F}}$	$\dfrac{p \vdash \mathbf{F}}{\sim p} \quad \dfrac{\sim p \vdash \mathbf{F}}{p}$
$\sim\sim$	$\dfrac{p}{\sim\sim p}$	$\dfrac{\sim\sim p}{p}$

R5 Quiz Answers

1. What is the contrapositive of $p \to q$? $\sim q \to \sim p$
2. What is the converse of $p \to q$? $q \to p$
3. What is the inverse of $p \to q$? $\sim p \to \sim q$

Do the following equivlalences hold?

4. $p \to q \equiv q \to p$ no
5. $p \to q \equiv \sim q \to \sim p$ yes
6. $q \to p \equiv \sim q \to \sim p$ no
7. $q \to p \equiv \sim p \to \sim q$ yes
8. $p \to q \equiv \sim p \vee q$ yes
9. $p \leftrightarrow q \equiv \sim p \oplus q$ yes
10. $\sim p \leftrightarrow q \equiv \sim (p \leftrightarrow q)$ yes

Match the following English Statements with their meaning in PropCalc.

11. p if q $q \rightarrow p$
12. p only if q $p \rightarrow q$
13. p if and only if q $p \leftrightarrow q$
14. p unless q $\sim q \rightarrow p$
15. r is a necessary condition for s to hold $s \rightarrow r$
16. r is a sufficient condition for s to hold $r \rightarrow s$
17. For s to hold, it is necessary and sufficient that r holds $r \leftrightarrow s$