CS250: Discrete Math for Computer Science

L4: PropCalc: Tautologies, Satisfiability, Equivalence

Definition of Propositional Connectives

via Truth Tables: **Today just concentrate on $\sim, \wedge, \vee$**

world	p	q	**T**	**F**	$\sim p$	$p \wedge q$	$p \vee q$	$p \oplus q$
W_3	1	1	1	0	0	1	1	0
W_2	1	0	1	0	0	0	1	1
W_1	0	1	1	0	1	0	1	1
W_0	0	0	1	0	1	0	0	0

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W_0	0	0	1	0	1	0	0	0

via functions: $\sim : \text{bool} \to \text{bool}; \quad \wedge, \vee, \oplus : \text{bool}^2 \to \text{bool} \qquad \text{bool} \stackrel{\text{def}}{=} \{0, 1\}$

$$\mathbf{T} \stackrel{\text{def}}{=} 1 \qquad \mathbf{F} \stackrel{\text{def}}{=} 0$$

$$\sim p \stackrel{\text{def}}{=} 1 - p \qquad p \wedge q \stackrel{\text{def}}{=} \min(p, q)$$

$$p \vee q \stackrel{\text{def}}{=} \max(p, q) \qquad p \to q \stackrel{\text{def}}{=} \sim p \vee q$$

$$p \leftrightarrow q \stackrel{\text{def}}{=} (p \to q) \wedge (q \to p) \qquad p \oplus q \stackrel{\text{def}}{=} (p + q) \textbf{ mod } 2$$

Definition of Propositional Connectives

via Truth Tables: **Today just concentrate on $\sim, \wedge, \vee$**

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W_1	0	1	1	0	1	0	1	1
W_0	0	0	1	0	1	0	0	0

Key Idea: $W_i : \{p_1, \ldots, p_n\} \to \mathrm{bool}$ $W_0, \ldots, W_{2^n-1}$

lines of truth table = valuations = possible worlds

via functions: $\sim : \mathrm{bool} \to \mathrm{bool}$; $\wedge, \vee, \oplus : \mathrm{bool}^2 \to \mathrm{bool}$ $\mathrm{bool} \stackrel{\mathrm{def}}{=} \{0, 1\}$

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possible worlds	lines of truth table	valuations
$W_0, \ldots, W_{2^n-1}$		$W_i : \{p_1, \ldots, p_n\} \to \mathrm{bool}$

world	p	q	**T**	**F**	$\sim p$	$p \wedge q$	$p \vee q$	$p \oplus q$
W_3	1	1	1	0	0	1	1	0
W_2	1	0	1	0	0	0	1	1
W_1	0	1	1	0	1	0	1	1
W_0	0	0	1	0	1	0	0	0

possible worlds $W_0, \ldots, W_{2^n-1}$ | lines of truth table | valuations $W_i : \{p_1, \ldots, p_n\} \to \mathrm{bool}$

world	p	q	**T**	**F**	$\sim p$	$p \wedge q$	$p \vee q$	$p \oplus q$
W_3	1	1	1	0	0	1	1	0
W_2	1	0	1	0	0	0	1	1
W_1	0	1	1	0	1	0	1	1
W_0	0	0	1	0	1	0	0	0

iClicker 4.1 $\sim p$ is true in which of the above worlds?

A: all of them
B: none of them
C: W_0 and W_1
D: W_2 and W_3

possible worlds	lines of truth table	valuations
$W_0, \ldots, W_{2^n-1}$		$W_i : \{p_1, \ldots, p_n\} \rightarrow \text{bool}$

world	p	q	**T**	**F**	$\sim p$	$p \wedge q$	$p \vee q$	$p \oplus q$
W_3	1	1	1	0	0	1	1	0
W_2	1	0	1	0	0	0	1	1
W_1	0	1	1	0	1	0	1	1
W_0	0	0	1	0	1	0	0	0

possible worlds $W_0, \ldots, W_{2^n-1}$ lines of truth table valuations $W_i : \{p_1, \ldots, p_n\} \to \text{bool}$

world	p	q	**T**	**F**	$\sim p$	$p \wedge q$	$p \vee q$	$p \oplus q$
W_3	1	1	1	0	0	1	1	0
W_2	1	0	1	0	0	0	1	1
W_1	0	1	1	0	1	0	1	1
W_0	0	0	1	0	1	0	0	0

iClicker 4.2 $(p \vee q) \wedge \sim(p \wedge q)$ is true in which of the above worlds?

A: none of them
B: W_1 and W_2
C: just W_3
D: just W_1

Tautologies, Contradictions, and Satisfiability

- A **tautology** (**Taut**) is a PropCalc formula such that every row of its truth table is 1, i.e., it is true in **all worlds**,

world	p	q	$\sim p$	$p \vee \sim p$	$p \wedge \sim p$	$p \wedge q$
W_3	1	1	0	1	0	1
W_2	1	0	0	1	0	0
W_1	0	1	1	1	0	0
W_0	0	0	1	1	0	0

Tautologies, Contradictions, and Satisfiability

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world	p	q	$\sim p$	**Taut** $p \vee \sim p$	$p \wedge \sim p$	$p \wedge q$
W_3	1	1	0	1	0	1
W_2	1	0	0	1	0	0
W_1	0	1	1	1	0	0
W_0	0	0	1	1	0	0

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- A **contradiction** (**unSAT**) is a PropCalc formula whose truth table is all 0's, i.e. it is true in **no world**,

world	p	q	$\sim p$	**Taut** $p \vee \sim p$	$p \wedge \sim p$	$p \wedge q$
W_3	1	1	0	1	0	1
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W_1	0	1	1	1	0	0
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W_3	1	1	0	1	0	1
W_2	1	0	0	1	0	0
W_1	0	1	1	1	0	0
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Tautologies, Contradictions, and Satisfiability

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- A **contradiction** (**unSAT**) is a PropCalc formula whose truth table is all 0's, i.e. it is true in **no world**, e.g., $p \wedge \sim p \in$ **unSAT**.
- A PropCalc formula is **satisfiable** (**SAT**) iff it is not a contradiction, i.e., it is true in **some world**,

world	p	q	$\sim p$	**Taut** $p \vee \sim p$	**unSAT** $p \wedge \sim p$	$p \wedge q$
W_3	1	1	0	1	0	1
W_2	1	0	0	1	0	0
W_1	0	1	1	1	0	0
W_0	0	0	1	1	0	0

Tautologies, Contradictions, and Satisfiability

- A **tautology** (**Taut**) is a PropCalc formula such that every row of its truth table is 1, i.e., it is true in **all worlds**, e.g., $p \vee \sim p \in$ **Taut**.
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- A PropCalc formula is **satisfiable** (**SAT**) iff it is not a contradiction, i.e., it is true in **some world**, e.g., p, q, $\sim p$, $p \vee \sim p$, $p \wedge q \in$ **SAT**.

world	**SAT** p	**SAT** q	**SAT** $\sim p$	**SAT** **Taut** $p \vee \sim p$	**unSAT** $p \wedge \sim p$	**SAT** $p \wedge q$
W_3	1	1	0	1	0	1
W_2	1	0	0	1	0	0
W_1	0	1	1	1	0	0
W_0	0	0	1	1	0	0

iClicker 4.3 $p \vee (\sim p \wedge q)$ is ?

A: **Taut**

B: **unSAT**

C: **neither**, i.e., **SAT** but not **Taut**

world	p	q	$\sim p$	$p \vee (\sim p \wedge q)$
W_3	1	1	0	
W_2	1	0	0	
W_1	0	1	1	
W_0	0	0	1	

R4 Quiz Answers

Is the following PropForm a tautology (**Taut**), a contradiction (**unSAT**), or neither (**SAT** - **Taut**) ?

1. $\sim\sim p \vee p$ **SAT** $-$ **Taut**

R4 Quiz Answers

Is the following PropForm a tautology (**Taut**), a contradiction (**unSAT**), or neither (**SAT** - **Taut**) ?

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2. $\sim\sim p \vee \sim p$ **Taut**

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1. $\sim\sim p \vee p$ **SAT** $-$ **Taut**
2. $\sim\sim p \vee \sim p$ **Taut**
3. $p \oplus p$ **unSAT**

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3. $p \oplus p$ **unSAT**
4. $p \oplus \sim p$ **Taut**

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2. $\sim\sim p \vee \sim p$ **Taut**
3. $p \oplus p$ **unSAT**
4. $p \oplus \sim p$ **Taut**
5. $p \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**

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Is the following PropForm a tautology (**Taut**), a contradiction (**unSAT**), or neither (**SAT** - **Taut**) ?

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2. $\sim\sim p \vee \sim p$ **Taut**
3. $p \oplus p$ **unSAT**
4. $p \oplus \sim p$ **Taut**
5. $p \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**
6. $(p \wedge q) \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**

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2. $\sim\sim p \vee \sim p$ **Taut**
3. $p \oplus p$ **unSAT**
4. $p \oplus \sim p$ **Taut**
5. $p \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**
6. $(p \wedge q) \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**
7. $(p \wedge q) \wedge (\sim p \vee \sim q)$ **unSAT**

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3. $p \oplus p$ **unSAT**
4. $p \oplus \sim p$ **Taut**
5. $p \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**
6. $(p \wedge q) \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**
7. $(p \wedge q) \wedge (\sim p \vee \sim q)$ **unSAT**
8. $(p \oplus q) \wedge (\mathbf{T} \wedge p) \wedge (\mathbf{F} \vee q)$ **unSAT**

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6. $(p \wedge q) \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**
7. $(p \wedge q) \wedge (\sim p \vee \sim q)$ **unSAT**
8. $(p \oplus q) \wedge (\mathbf{T} \wedge p) \wedge (\mathbf{F} \vee q)$ **unSAT**
9. $(p \oplus q) \vee (p \wedge q) \vee (\sim p \vee \sim q)$ **Taut**

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2. $\sim\sim p \vee \sim p$ **Taut**
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5. $p \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**
6. $(p \wedge q) \wedge (\sim p \vee q)$ **SAT** $-$ **Taut**
7. $(p \wedge q) \wedge (\sim p \vee \sim q)$ **unSAT**
8. $(p \oplus q) \wedge (\mathbf{T} \wedge p) \wedge (\mathbf{F} \vee q)$ **unSAT**
9. $(p \oplus q) \vee (p \wedge q) \vee (\sim p \vee \sim q)$ **Taut**
10. $(p \wedge q) \vee (\sim p \vee \sim q)$ **Taut**

Knights and Knaves [Smullyan, *What Is the Name of This Book?*]

Knights always truthful; Knaves always lie; $A, B \in \{\text{Kt}, \text{Kv}\}$

B : "$A \& B$ opposite types"

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$T_1 \stackrel{\text{def}}{=}$ B is a Kt $T_2 \stackrel{\text{def}}{=}$ $A \& B$ opposite types

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$$T_1 \stackrel{\text{def}}{=} B \text{ is a Kt} \qquad T_2 \stackrel{\text{def}}{=} A\&B \text{ opposite types}$$

$$S_1 = T_1 \leftrightarrow A \text{ is Kt} \qquad S_2 = T_2 \leftrightarrow B \text{ is Kt}$$

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$$S_1 = T_1 \leftrightarrow A \text{ is Kt} \qquad S_2 = T_2 \leftrightarrow B \text{ is Kt}$$

w	A is Kt	B is Kt	T_1	T_2	$T_1 \leftrightarrow A$ is Kt	$T_2 \leftrightarrow B$ is Kt
W_3	1	1	1	0	1	0
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W_0	0	0	0	0	1	1

W_0 is only world satisfying $S_1 \wedge S_2$.

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$$S_1 = T_1 \leftrightarrow A \text{ is Kt} \qquad S_2 = T_2 \leftrightarrow B \text{ is Kt}$$

w	A is Kt	B is Kt	T_1	T_2	$T_1 \leftrightarrow A$ is Kt	$T_2 \leftrightarrow B$ is Kt
W_3	1	1	1	0	1	0
W_2	1	0	0	1	0	0
W_1	0	1	1	1	0	1
W_0	0	0	0	0	1	1

W_0 is only world satisfying $S_1 \wedge S_2$.

Thus *A* and *B* are both Knaves.

R4 Quiz Answers

Do the following equivalences hold?

1. $\mathbf{T} \equiv p \oplus \sim p$ yes

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1. $\mathbf{T} \equiv p \oplus \sim p$ yes
2. $q \wedge p \equiv p \wedge (\sim p \vee q)$ yes

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2. $q \wedge p \equiv p \wedge (\sim p \vee q)$ yes
3. $\sim p \vee q \equiv q \vee p$ no

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2. $q \wedge p \equiv p \wedge (\sim p \vee q)$ yes
3. $\sim p \vee q \equiv q \vee p$ no
4. $\sim p \vee q \equiv \sim q \vee p$ no

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1. $\mathbf{T} \equiv p \oplus \sim p$ yes
2. $q \wedge p \equiv p \wedge (\sim p \vee q)$ yes
3. $\sim p \vee q \equiv q \vee p$ no
4. $\sim p \vee q \equiv \sim q \vee p$ no
5. $(p \wedge \sim q) \equiv \sim(\sim p \vee q)$ yes