CS250: Discrete Math for Computer Science

L33: Kleene's Theorem

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$$\delta^\star(\{s\}, wa) = \delta(\delta^\star(\{s\}, w), a) = \delta(\Delta^\star(s, w), a)$$

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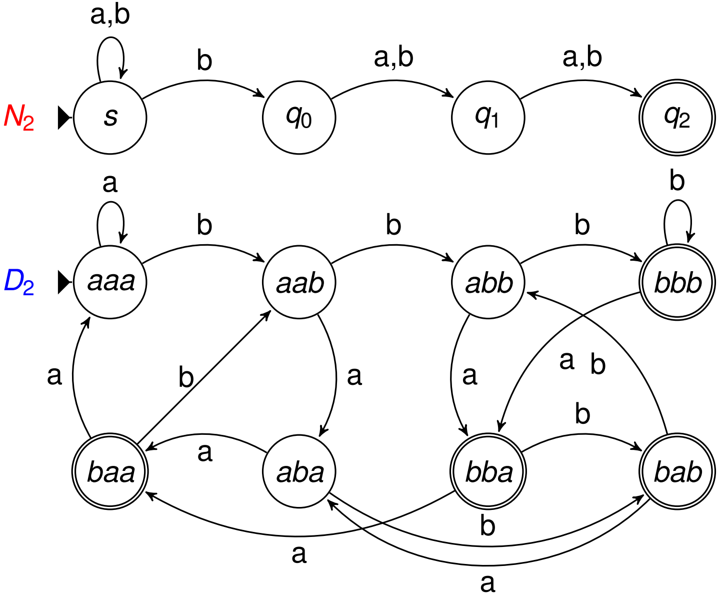
$$\begin{aligned} \delta^\star(\{s\}, wa) &= \delta(\delta^\star(\{s\}, w), a) = \delta(\Delta^\star(s, w), a) \\ &= \bigcup_{q \in \Delta^\star(s,w)} \Delta(q, a) = \Delta^\star(s, wa) \qquad \square \end{aligned}$$

$\mathcal{L}(N_2) = \mathcal{L}(D_2)$



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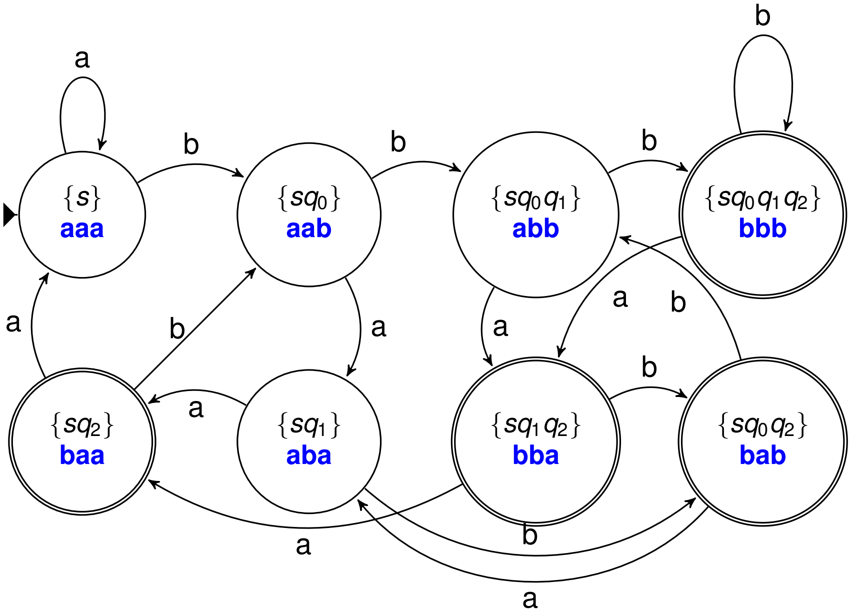

Kleene's Theorem Let $A \subseteq \Sigma^\star$ be any language. Then the following are equivalent:

1. $A = \mathcal{L}(D)$, for some DFA D.
2. $A = \mathcal{L}(N)$, for some NFA N wo ϵ transitions.
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Now, we will prove: $(4) \Rightarrow (3) \ \wedge \ (1) \Rightarrow (4)$

(4) $\Rightarrow$ (3): $\forall e \in \mathrm{regexp}(\Sigma)\ \exists$ NFA N $(\mathcal{L}(e) = \mathcal{L}(N))$.

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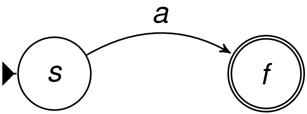

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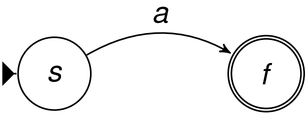
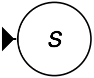


$e = \emptyset$



inductive cases: Assume **indHyp:** $\mathcal{L}(N_i) = \mathcal{L}(e_i),\ i = 1, 2$

$$N_i \quad = \quad (Q_i, \Sigma, \Delta_i, s_i, F_i) \qquad Q_1 \cap Q_2 = \emptyset$$

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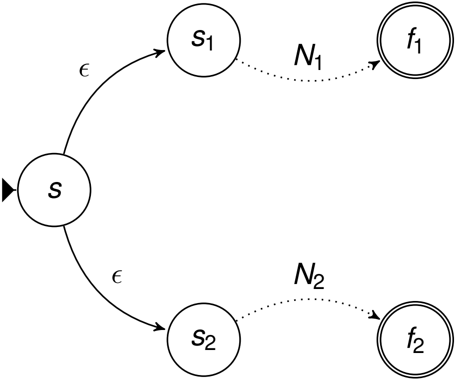
$$N_{e_1|e_2} \quad = \quad (\{s\} \cup Q_1 \cup Q_2, \Sigma, \{(s, \epsilon, \{s_1, s_2\})\} \cup \Delta_1 \cup \Delta_2, s, F_1 \cup F_2)$$

$N_{e_1|e_2}$



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$$e = e_1|e_2 \qquad \mathcal{L}(N_{e_1|e_2}) = \mathcal{L}(N_1) \cup \mathcal{L}(N_2) = \mathcal{L}(e)$$

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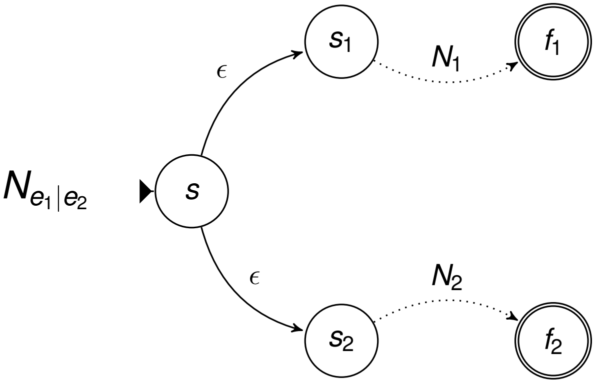

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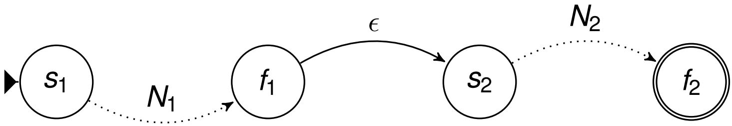

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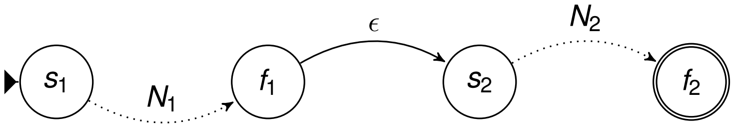

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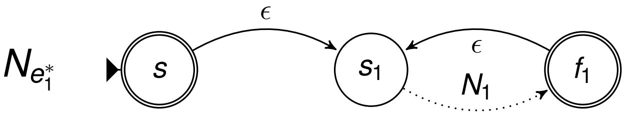
$$N_{e_1^*} = (\{s\} \cup Q_1, \Sigma, \{(s, \epsilon, \{s_1\})\} \cup \Big\{(f_1, \epsilon, \{s_1\}) \ \Big| \ f_1 \in F_1\Big\} \cup \Delta_1, s, \{s\} \cup F_1)$$



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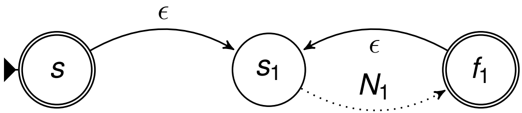

This completes the proof of $(4) \Rightarrow (3)$**.**

(1) $\Rightarrow$ (4): **Assume:** $A = \mathcal{L}(D)$, $D = (Q, \Sigma, \delta, s, F)$ where $Q = \{1, \ldots, n\}$, $F = \{f_1, \ldots, f_r\}$.

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Use **dynamic programming algorithm** to construct $e \in \mathrm{regexp}(\Sigma)$ s.t. $\mathcal{L}(e) = \mathcal{L}(D)$. Build regexps e_{ij}^{k} s.t.

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Goal : $\mathcal{L}(e_{ij}^k) = \{w \mid i \overset{\star}{\underset{w}{\to}} j;$ no intermediate state $\# > k\}$

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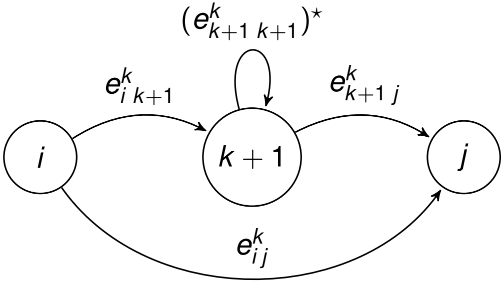
base case $e_{ij}^0 := \{a \mid j = \delta(i, a)\} \mid \{\epsilon \mid i = j\} \mid \emptyset$

inductive case $e_{ij}^{k+1} := e_{ij}^k \mid e_{i\,k+1}^k \, (e_{k+1\,k+1}^k)^\star \, e_{k+1\,j}^k$



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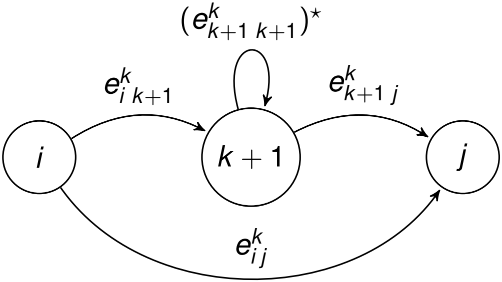

$e = e_{s\,f_1}^n | e_{s\,f_2}^n | \cdots | e_{s\,f_r}^n$

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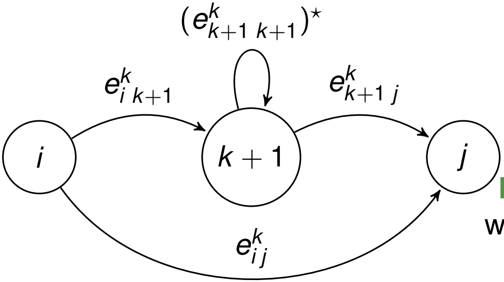

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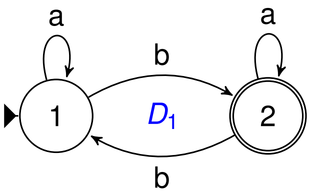
Note: this construction works for NFAs as well.

Goal :	$\mathcal{L}(e_{ij}^k)$	$=$	$\{ w \mid i \xrightarrow[w]{\star} j;\ \text{no intermediate state } \# > k \}$
base case	e_{ij}^0	$:=$	$\{ a \mid j = \delta(i,a) \} \mid \{ \epsilon \mid i = j \} \mid \emptyset$
inductive case	e_{ij}^{k+1}	$:=$	$e_{ij}^k \mid e_{i\,k+1}^k \,(e_{k+1\,k+1}^k)^\star\, e_{k+1\,j}^k$



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base case $e_{ij}^0 := \{ a \mid j = \delta(i, a) \} \mid \{ \epsilon \mid i = j \} \mid \emptyset$

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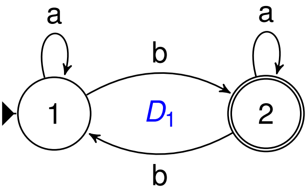

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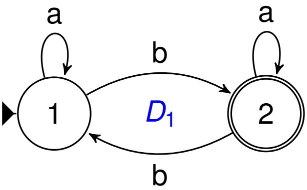
What is e_{12}^0 ?

A: a **B:** b **C:** ϵ **D:** $\emptyset$

Goal : $\mathcal{L}(e^k_{ij}) = \{ w \mid i \xrightarrow[w]{\star} j;\ \text{no intermediate state } \# > k \}$

base case $e^0_{ij} := \{ a \mid j = \delta(i,a) \} \mid \{ \epsilon \mid i = j \} \mid \emptyset$

inductive case $e^{k+1}_{ij} := e^k_{ij} \mid e^k_{i\,k+1}\ (e^k_{k+1\,k+1})^\star\ e^k_{k+1\,j}$

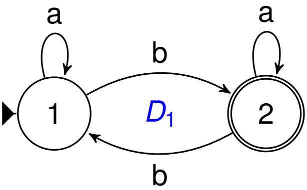


$$e^0 = \begin{pmatrix} a|\epsilon & b \\ b & a|\epsilon \end{pmatrix}$$

Goal : $\mathcal{L}(e_{ij}^k) = \{ w \mid i \xrightarrow[w]{\star} j;$ no intermediate state $\# > k \}$

base case $e_{ij}^0 := \{ a \mid j = \delta(i,a) \} \mid \{ \epsilon \mid i = j \} \mid \emptyset$

inductive case $e_{ij}^{k+1} := e_{ij}^k \mid e_{i\,k+1}^k \, (e_{k+1\,k+1}^k)^\star \, e_{k+1\,j}^k$



$$e^0 = \begin{pmatrix} a|\epsilon & b \\ b & a|\epsilon \end{pmatrix}$$

iClicker 33.2

What is e_{12}^1 ?

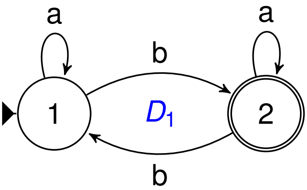
A: a **B:** b

C: $a^* b$ **D:** $a^* b a^*$

Goal : $\mathcal{L}(e^k_{ij}) = \{w \mid i \overset{\star}{\underset{w}{\to}} j;\ \text{no intermediate state } \# > k\}$

base case $e^0_{ij} := \{a \mid j = \delta(i,a)\} \mid \{\epsilon \mid i = j\} \mid \emptyset$

inductive case $e^{k+1}_{ij} := e^k_{ij} \mid e^k_{i\,k+1}\,(e^k_{k+1\,k+1})^\star\, e^k_{k+1\,j}$



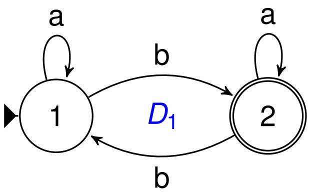
$$e^0 = \begin{pmatrix} a|\epsilon & b \\ b & a|\epsilon \end{pmatrix}$$

$$e^1 = \begin{pmatrix} a^* & a^*b \\ ba^* & a|\epsilon|ba^*b \end{pmatrix}$$

Goal : $\mathcal{L}(e^k_{ij}) \;=\; \{ w \mid i \overset{\star}{\underset{w}{\to}} j;\ \text{no intermediate state} \# > k \}$

base case $e^0_{ij} \;:=\; \{ a \mid j = \delta(i,a) \} \mid \{ \epsilon \mid i = j \} \mid \emptyset$

inductive case $e^{k+1}_{ij} \;:=\; e^k_{ij} \mid e^k_{i\,k+1}\,(e^k_{k+1\,k+1})^\star\, e^k_{k+1\,j}$



$$e^0 = \begin{pmatrix} a|\epsilon & b \\ b & a|\epsilon \end{pmatrix}$$

$$e^1 = \begin{pmatrix} a^* & a^*b \\ ba^* & a|\epsilon|ba^*b \end{pmatrix}$$

iClicker 33.3

What is e^2_{12} ?

A: a^*ba^*

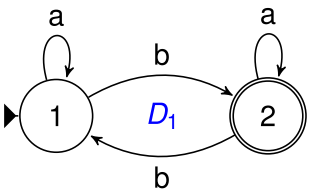
B: $a^*b(a|ba^*b)^*$

C: $a^*b|(a^*ba^*ba^*)^*$

Goal : $\mathcal{L}(e_{ij}^k) = \{w \mid i \xrightarrow[w]{\star} j;\ \text{no intermediate state } \# > k\}$

base case $e_{ij}^0 := \{a \mid j = \delta(i,a)\} \mid \{\epsilon \mid i = j\} \mid \emptyset$

inductive case $e_{ij}^{k+1} := e_{ij}^k \mid e_{i\,k+1}^k\ (e_{k+1\,k+1}^k)^\star\ e_{k+1\,j}^k$



$$e^0 = \begin{pmatrix} a|\epsilon & b \\ b & a|\epsilon \end{pmatrix}$$

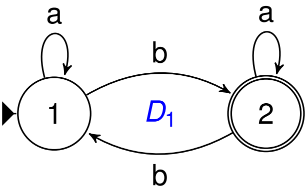
$$e^1 = \begin{pmatrix} a^* & a^*b \\ ba^* & a|\epsilon|ba^*b \end{pmatrix}$$

$$e^2 = \begin{pmatrix} (a|ba^*b)^* & a^*b(a|ba^*b)^* \\ a^*b(a|ba^*b)^* & (a|ba^*b)^* \end{pmatrix}$$

Goal : $\mathcal{L}(e_{ij}^{k}) = \{w \mid i \xrightarrow[w]{\star} j;\ \text{no intermediate state } \# > k\}$

base case $e_{ij}^{0} := \{a \mid j = \delta(i,a)\} \mid \{\epsilon \mid i = j\} \mid \emptyset$

inductive case $e_{ij}^{k+1} := e_{ij}^{k} \mid e_{i\,k+1}^{k}\ (e_{k+1\,k+1}^{k})^{\star}\ e_{k+1\,j}^{k}$



$$e^0 = \begin{pmatrix} a|\epsilon & b \\ b & a|\epsilon \end{pmatrix}$$

$$e^1 = \begin{pmatrix} a^* & a^*b \\ ba^* & a|\epsilon|ba^*b \end{pmatrix}$$

$$e^2 = \begin{pmatrix} (a|ba^*b)^* & a^*b(a|ba^*b)^* \\ a^*b(a|ba^*b)^* & (a|ba^*b)^* \end{pmatrix}$$

$$\mathcal{L}(D_1) = \mathcal{L}(e_{12}^2) = \mathcal{L}(a^*b(a|ba^*b)^*)$$

Kleene's Theorem Let $A \subseteq \Sigma^\star$ be any language. Then the following are equivalent:

1. $A = \mathcal{L}(D)$, for some DFA D.
2. $A = \mathcal{L}(N)$, for some NFA N wo ϵ transitions.
3. $A = \mathcal{L}(N)$, for some NFA N.
4. $A = \mathcal{L}(e)$, for some regular expression e.
5. A is regular.

We have completed the proof of Kleene's Theorem! $\square$