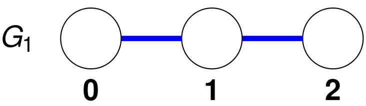
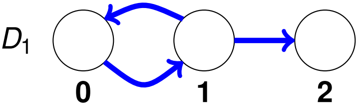
CS250: Discrete Math for Computer Science

L28: Searching Undirected Graphs: Depth First Search

Walks versus Paths

A **path** is a walk that **never visits the same edge or vertex twice**, **except** a path may **start and end** at the **same vertex** in which case it is called a **cycle**.





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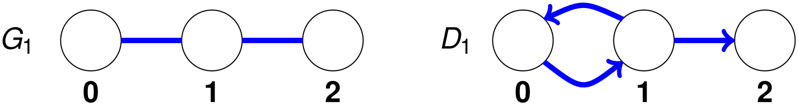

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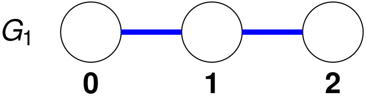
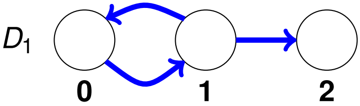
Note: this definition differs for directed and undirected graphs. The undirected graph G_1 is acyclic.



acyclic



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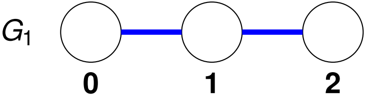
Note: this definition differs for directed and undirected graphs. The undirected graph G_1 is acyclic.
However, the directed graph, D_1, has a cycle: $(0, 1, 0)$.



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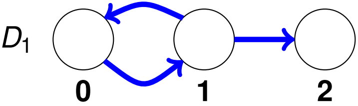
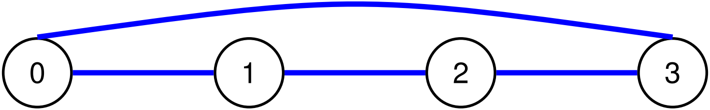
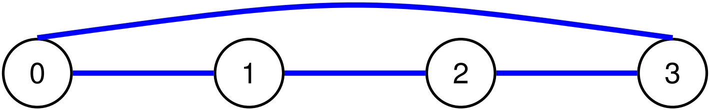


cyclic



$w_1 = (0)$ $w_2 = (0, 1)$ $w_3 = (0, 1, 2)$

$w_4 = (0, 1, 0)$ $w_5 = (0, 1, 2, 3)$ $w_6 = (0, 1, 2, 3, 0)$



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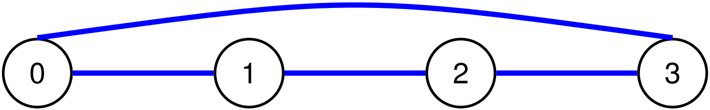
iClicker 28.1 In the above undirected graph, which of the above walks are paths?

A: all of them **B: all except w_4** **C: all except w_4 and w_6**



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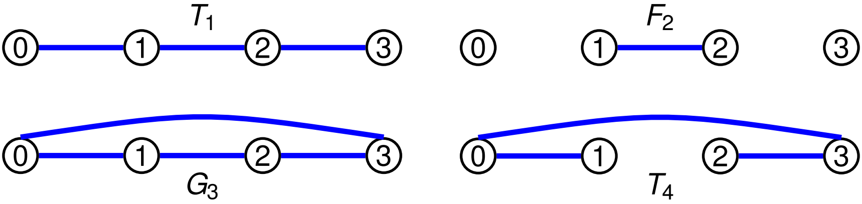
iClicker 28.2 Which of the above walks are cycles?

A: w_4 and w_6 **B: just w_6**

Cyclic versus Acyclic

A graph with at **at least one cycle** is called **cyclic**.



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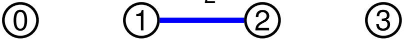
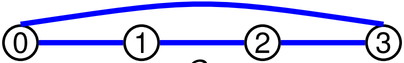
A graph that has **no cycles** is called **acyclic**.

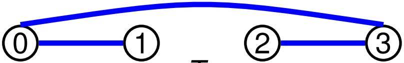
T_1



F_2



G_3



T_4

Cyclic versus Acyclic

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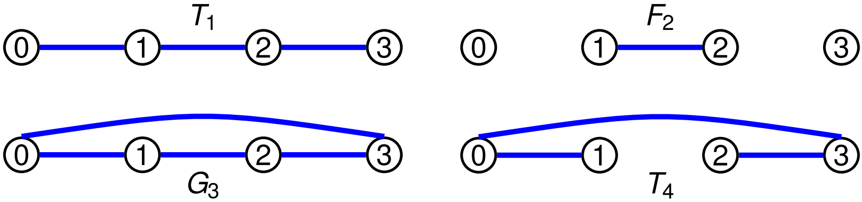

Rest of today: all graphs will be **undirected** and **loop free**.

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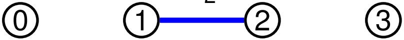
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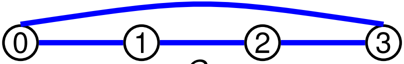
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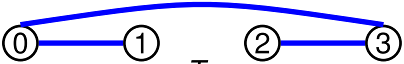
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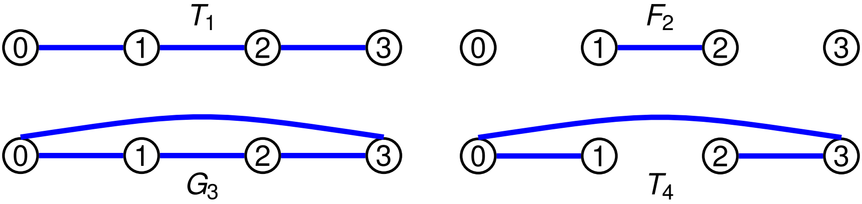

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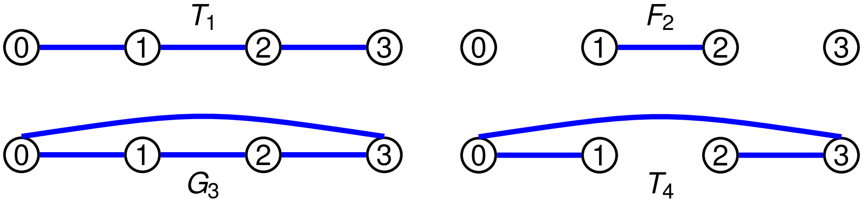

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A **forest** is an **undirected acyclic graph**.

An undirected graph is **connected** if every pair of vertices has a path between them.

An undirected **tree** is a **connected forest**.

Connected Components

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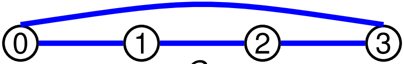

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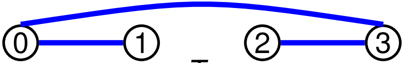

T_4

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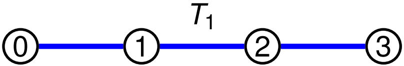

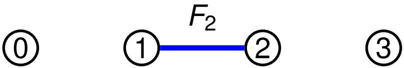

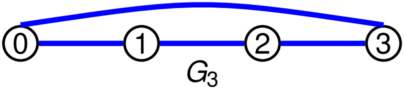

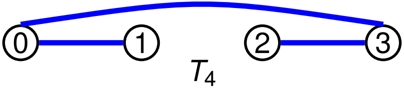

iClicker 28.3 How many connected components does F_2 have? $V^{F_2} = \{0, 1, 2, 3\}$ $E^{F_2} = \{(1, 2), (2, 1)\}$

A: 1 **B:** 2 **C:** 3

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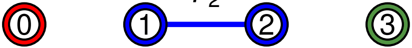
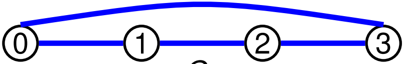
F_2





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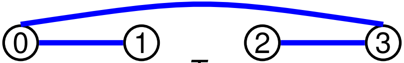

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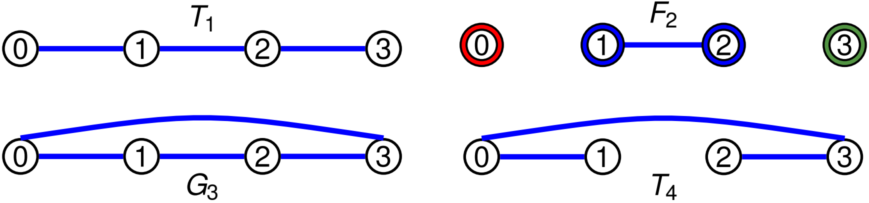

Recall E^* is the reflexive transitive closure of E. For undirected graphs, E^* is an equivalence relation and it partitions the vertices into connected components:

$$[v]_{E^*} \;=\; \left\{ u \in V \;\middle|\; E^*(u,v) \right\}$$

Wanted: fast algorithm to

- **search** a **graph**

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- **search** a **graph**
- compute its **connected components**
- determine if it is **cyclic** or **acyclic**

Depth First Search (undirected graphs)

```
DFSmain(G)
 for each u in V :
   color[u] = white
   parent[u] = NULL
 time=0
 for each u in V :
   if (color[u] == white):
     DFSVisit(u)
```

```
DFSVisit(u)
  color[u] = red                    // in process
  d[u] = ++time                     // discover
  for each v in Adj(u) :
    if (color[v] == white) :        // unseen
      parent[v] = u                 // tree edge
      DFSVisit(v)
  color[u] = black                  // done
  f[u] = ++time                     // finish
```

black Tree edge brown Back edge

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DFSVisit(u)
  color[u] = red
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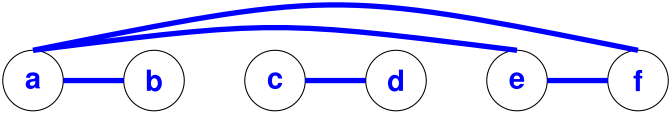

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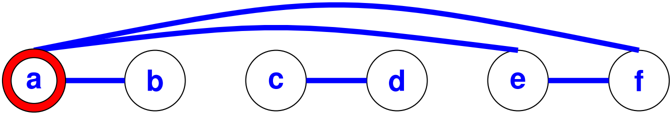

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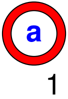

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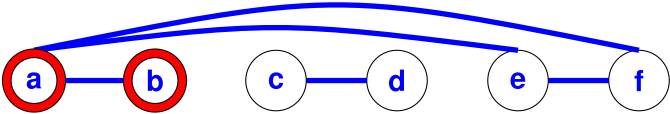

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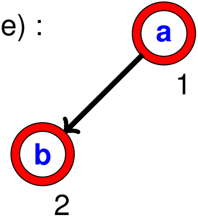

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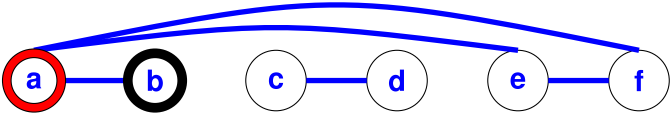

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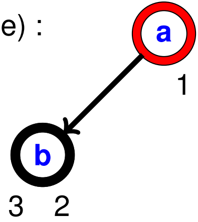

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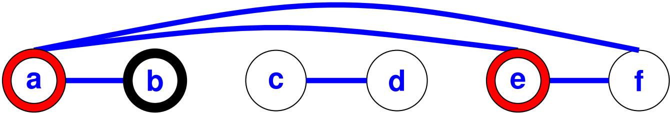

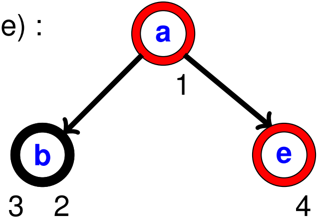

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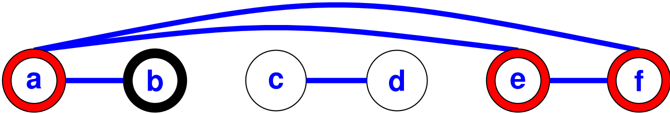

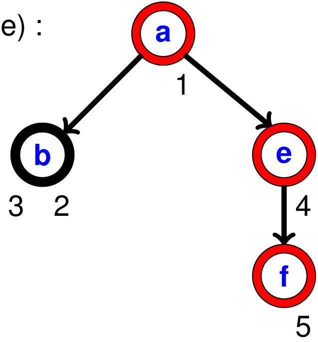

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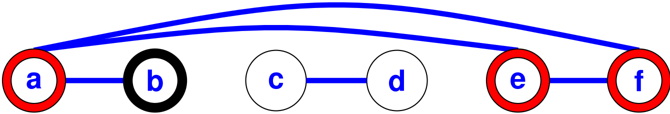

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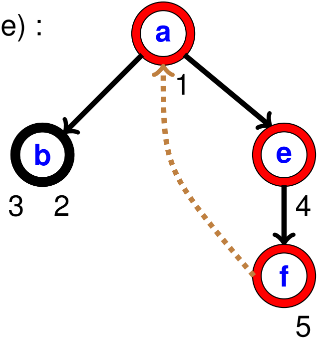

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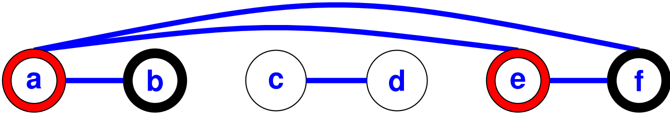

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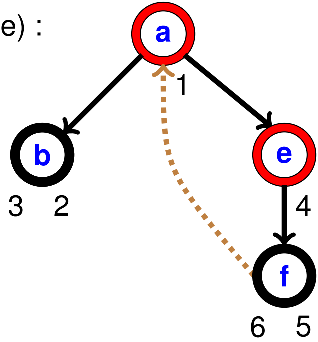

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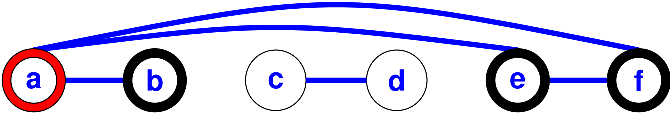

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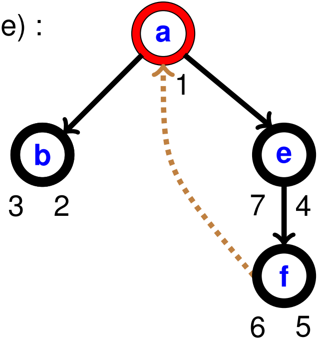

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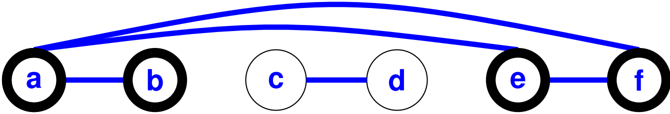

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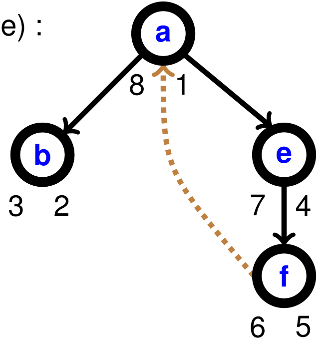

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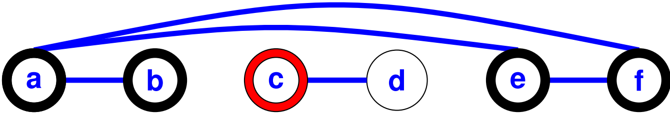
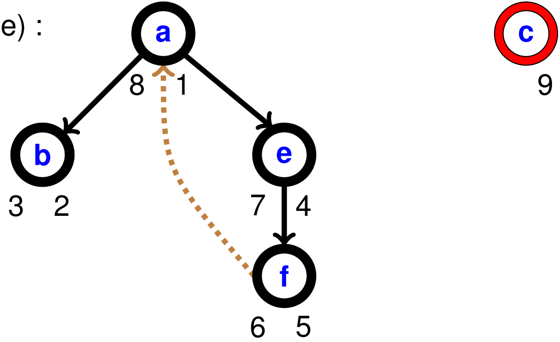
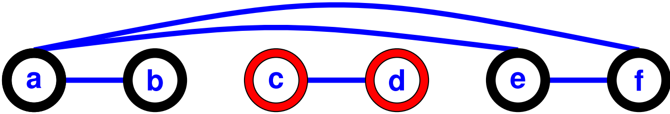
 for each v in Adj(u) :
   if (color[v] == white) :
     parent[v] = u
     DFSVisit(v)
 color[u] = black
 f[u] = ++time
```





black Tree edge brown Back edge



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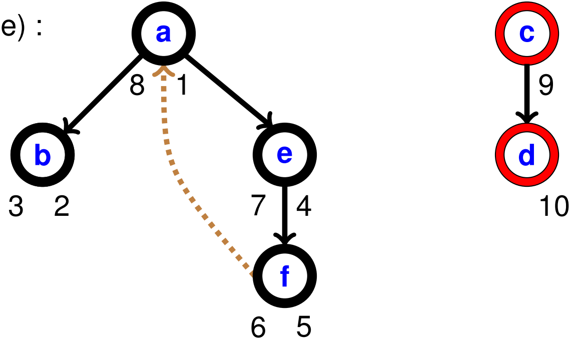

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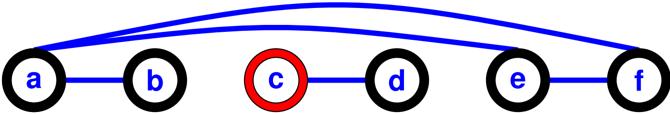

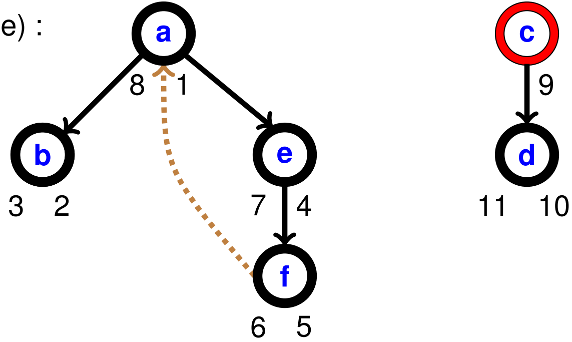

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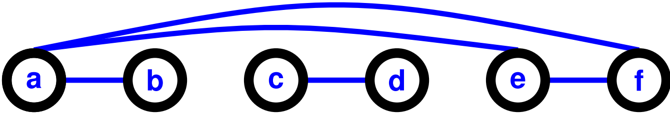

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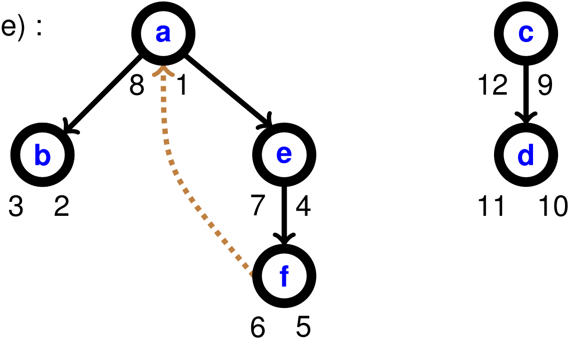

Depth First Search (undirected graphs)

```
DFSmain(G)
 for each u in V :
   color[u] = white
   parent[u] = NULL
 time=0
 for each u in V :
   if (color[u] == white):
     DFSVisit(u)
```

```
DFSVisit(u)
 color[u] = red                   // in process
 d[u] = ++time                    // discover
 for each v in Adj(u) :
   if (color[v] == white) :       // unseen
     parent[v] = u                // tree edge
     DFSVisit(v)
 color[u] = black                 // done
 f[u] = ++time                    // finish
```

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Let G be an undirected graph with n vertices and m edges.
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3. DFS determines which of these components is cyclic: a component is **cyclic** iff it has a **backedge**.

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base case: $1 = |[r]| = \{r\}$ visited first step of DFSVisit(r). ✓

inductive case: Assume **indHyp**: for all $H, r \in V^H$,
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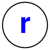

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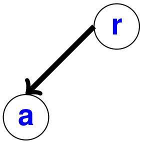

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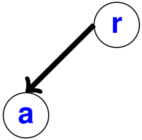
$H_1 \overset{\text{def}}{=}$ vertices reachable from a without going through r.



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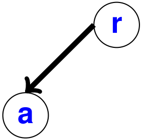

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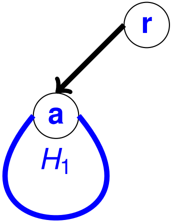

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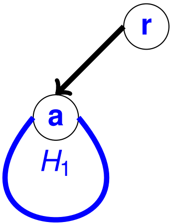

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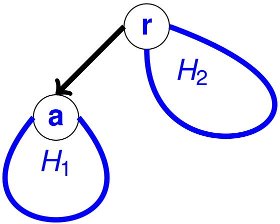

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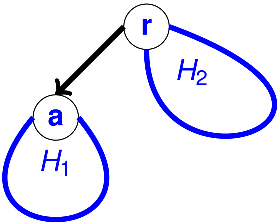

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By **indHyp**, DFSVisit(r) visits all of H_2.

Thus initial call of DFSVisit(r) visits all of $G = H_1 \cup H_2$. ✓

3. **Claim** *G* is cyclic iff DFS(*G*) finds a backedge.

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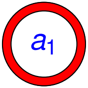
If there is a backege from *b* to *a*, then there is a path in the tree from *a* down to *b*, so this path plus the backedge forms a cycle. ✓

3. **Claim** G is cyclic iff DFS(G) finds a backedge.

If there is a backege from b to a, then there is a path in the tree from a down to b, so this path plus the backedge forms a cycle. ✓



Conversely, suppose that G contains a cycle and let $C = a_1, a_2, \ldots, a_{k-1}, a_1$ be a cycle. Let a_1 be the first vertex of the cycle that is visited in DFS(G).

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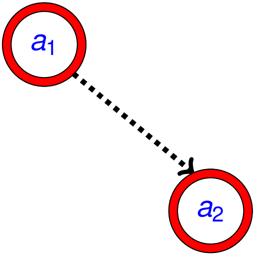

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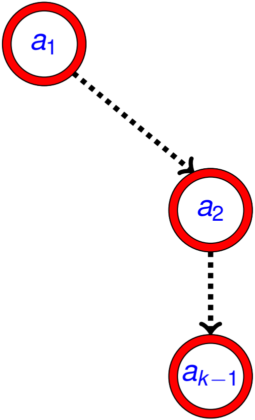

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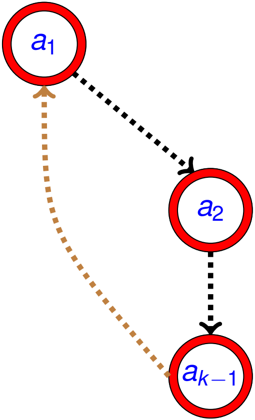

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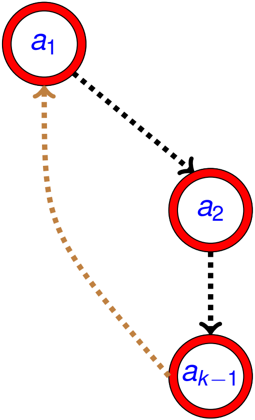

If there is a backege from b to a, then there is a path in the tree from a down to b, so this path plus the backedge forms a cycle. ✓

Conversely, suppose that G contains a cycle and let $C = a_1, a_2, \ldots, a_{k-1}, a_1$ be a cycle. Let a_1 be the first vertex of the cycle that is visited in DFS(G). By (2), we know that a_2 and a_{k-1} are visited during the call of DFSVisit(a_1). Assume that a_2 is visited first, the other case is symmetrical.

By (2), we know that a_{k-1} is visited during the call of DFSVisit(a_2).

When a_{k-1} is visited, a_1 is red and not a_{k-1}'s parent. Thus the edge (a_{k-1}, a_1) is a backedge. □