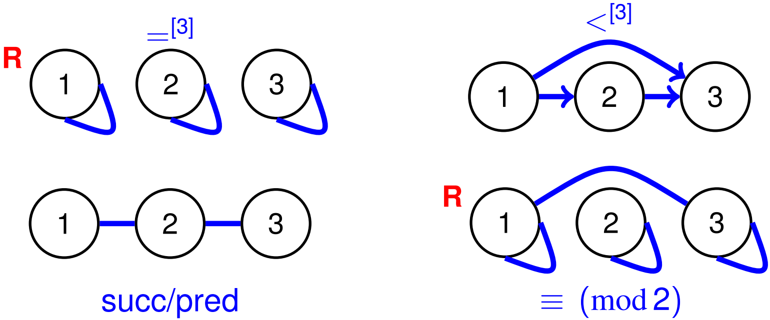
CS250: Discrete Math for Computer Science

L26: Equivalence Relations

Equivalence Relations

Reflexive $\equiv \forall x\, E(x,x)$

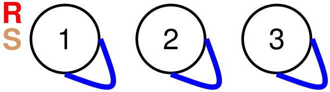


Equivalence Relations

$$\textbf{Reflexive} \equiv \forall x\, E(x,x)$$

$$\textbf{Symmetric} \equiv \forall xy\, (E(x,y) \rightarrow E(y,x))$$

$=^{[3]}$



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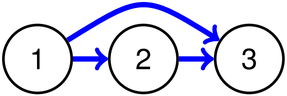
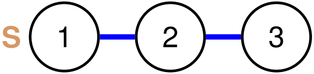
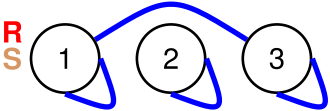




succ/pred



$\equiv \pmod 2$

Equivalence Relations

$$\textbf{Reflexive} \equiv \forall x\, E(x,x)$$
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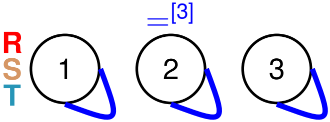

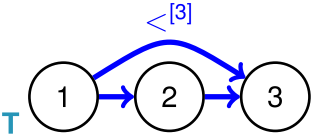

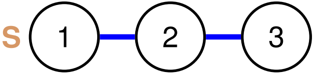

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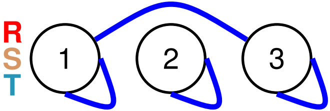

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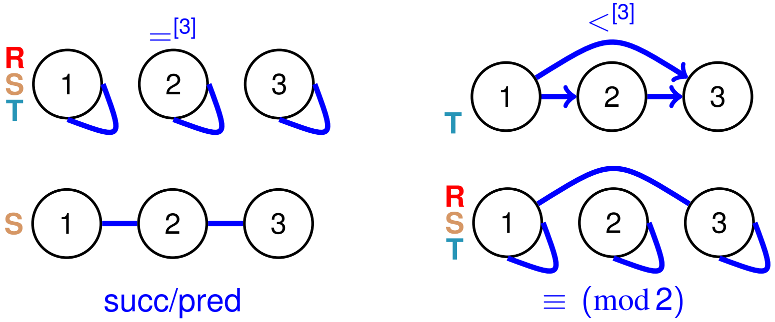

Def. An **equivalence relation** is a relation that is **reflexive**, **symmetric** and **transitive**.

Equivalence Relations on 3-Element Sets

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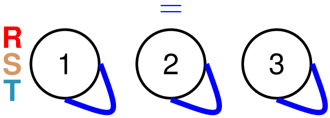

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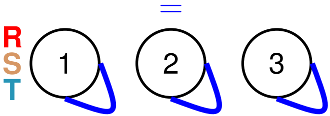

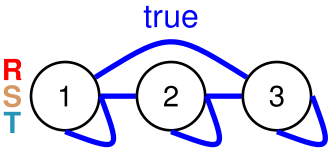

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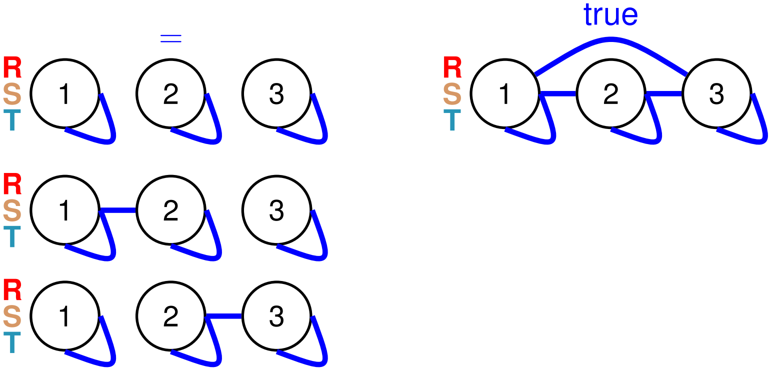

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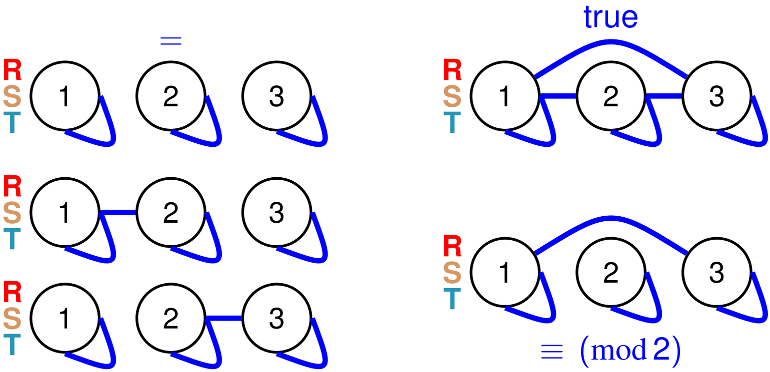

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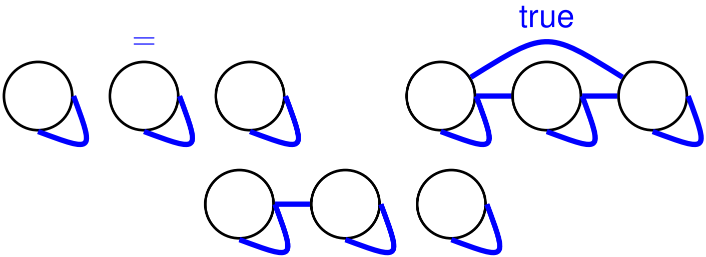

Partitions

Def. A **partition** of a non-empty set V is collection of pairwise disjoint, non-empty subsets, $(P_1, P_2, \ldots)$ of V whose union is V:

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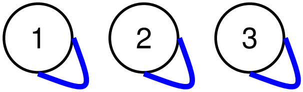

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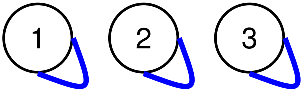

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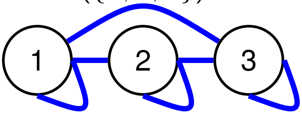

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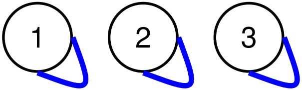

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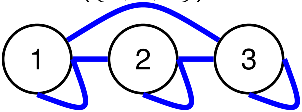

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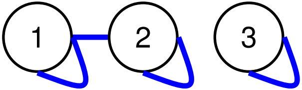

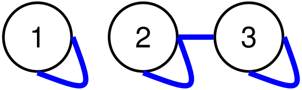

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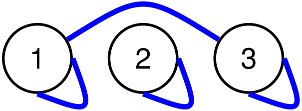

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Partitions relate to Equivalence Relations, How?

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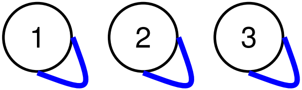

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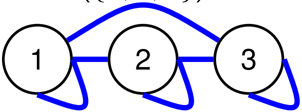

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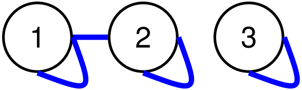

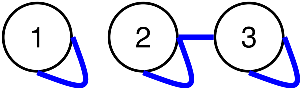

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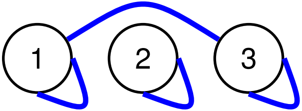

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Observe that the set of distinct equivalence classes, $([v]_\equiv), v \in V$, is a partition. □

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3. $a^n \equiv (a')^n \pmod{m}$

Recall L17: $a^{-1} \bmod m$ exists iff $\gcd(a, m) = 1$

$\cdot \mathbb{Z}/6\mathbb{Z}$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

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4	0	4	2	0	4	2
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$$|\mathbf{Z}_m^*| = \{a \in \mathbf{Z}/m\mathbf{Z} \mid \gcd(a, m) = 1\} \qquad x \cdot_{Z_m^*} y = (x \cdot y) \,\%\, m$$

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$$|\mathbf{Z}_6^*| = \{1, 5\}$$

$\cdot_{Z_6^*}$	1	5
1	1	5
5	5	1

$\mathbf{Z}_m^*$ is the multiplicative group mod m

$$|\mathbf{Z}_5^\star| = \{1, 2, 3, 4\}$$

$\cdot Z_5^\star$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

Groups

$$\Sigma_{\mathsf{group}} \stackrel{\text{def}}{=} (;\mathsf{1}, \cdot[\text{infix}]^{\mathsf{2}}, {}^{-1}[\text{postfix}]^{\mathsf{1}})$$

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$$
\begin{array}{lcll}
G_1 & = & \forall x\, y\, z \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z & \textit{associative} \\
G_2 & = & \forall x \quad x \cdot 1 = x & \textit{identity} \\
G_3 & = & \forall x \quad x \cdot x^{-1} = 1 & \textit{inverses}
\end{array}
$$

Def. A **group** is a $G \in \text{World}[\Sigma_{\text{group}}]$ s.t. $G \models G_1 \wedge G_2 \wedge G_3$.

Groups

$$\Sigma_{\text{group}} \stackrel{\text{def}}{=} (;1, \cdot[\text{infix}]^2, ^{-1}[\text{postfix}]^1)$$

$$\begin{array}{lcll} G_1 & = & \forall x\, y\, z \quad x \cdot (y \cdot z) = (x \cdot y) \cdot z & \textit{associative} \\ G_2 & = & \forall x \quad x \cdot 1 = x & \textit{identity} \\ G_3 & = & \forall x \quad x \cdot x^{-1} = 1 & \textit{inverses} \end{array}$$

Def. A **group** is a $G \in \text{World}[\Sigma_{\text{group}}]$ s.t. $G \models G_1 \wedge G_2 \wedge G_3$.

Prop. For all $m > 1$, Z_m^* is a group.

Euler's phi function, φ

Def. For $m > 1$,

$$\varphi(m) \stackrel{\text{def}}{=} \|\mathbf{Z}_m^*\| = \left|\{a \in \mathbf{Z}/m\mathbf{Z} \mid \gcd(a, m) = 1\}\right|$$

m	$\varphi(m)$	$\lvert\mathbf{Z}_m^*\rvert$
2	1	$\{1\}$
3	2	$\{1, 2\}$
4	2	$\{1, 3\}$
5	4	$\{1, 2, 3, 4\}$
6	2	$\{1, 5\}$
7	6	$\{1, 2, 3, 4, 5, 6\}$
8	4	$\{1, 3, 5, 7\}$
9	6	$\{1, 2, 4, 5, 7, 8\}$
10	4	$\{1, 3, 7, 9\}$
11	10	$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
12	4	$\{1, 5, 7, 11\}$

Fermat's Little Theorem

Thm: For p prime, $a \in \mathbf{Z}_p^\star$, $a^{p-1} \equiv 1 \pmod{p}$

Proof:

$f_a : \mathbf{Z}_p^\star \underset{\text{onto}}{\overset{1:1}{\rightarrow}} \mathbf{Z}_p^\star$; $\quad f_a(x) = (a \cdot x) \quad f_a^{-1}(x) = ((a^{-1} (\bmod p)) \cdot x)$

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$$\prod_{i \in \mathbf{Z}_p^\star} i \equiv \prod_{i \in \mathbf{Z}_p^\star} a \cdot i \pmod p$$

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$$1 \equiv a^{p-1} \pmod{p} \qquad \square$$