CS250: Discrete Math for Computer Science

L24: Functions

Functions Review

We defined **function** back in L3. Now, we will review what we know and improve our knowledge and understanding about functions.

Binary Relations

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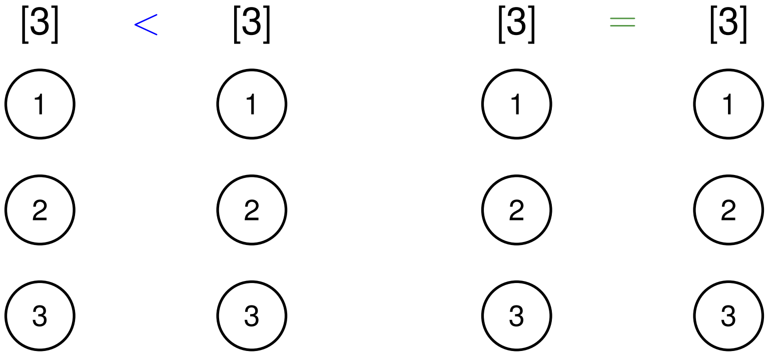
- **We'll finally talk about this today.**

Arrow Diagram of a Relation

For relation R from A to B, draw an arrow from a to b iff aRb.



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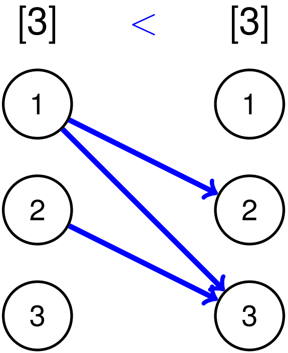
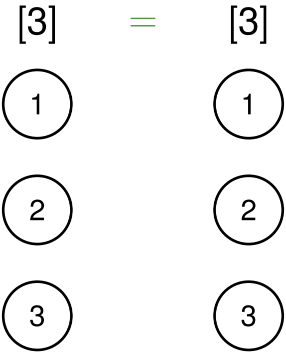


$\{(1,2),(1,3),(2,3)\}$



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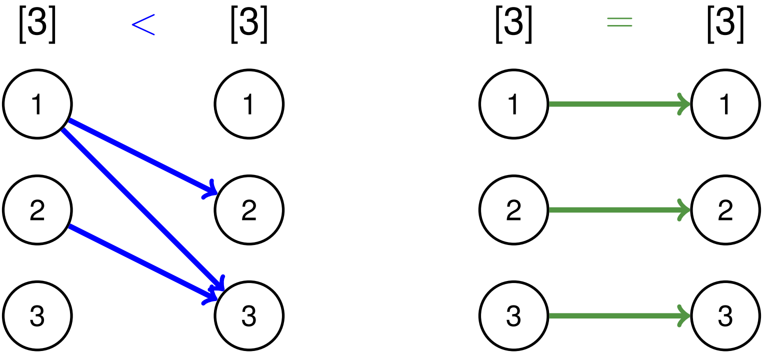

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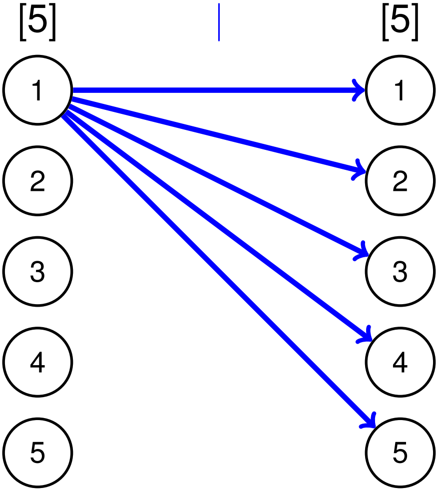

$\{(1,1),(1,2),(1,3),(1,4),(1,5),$

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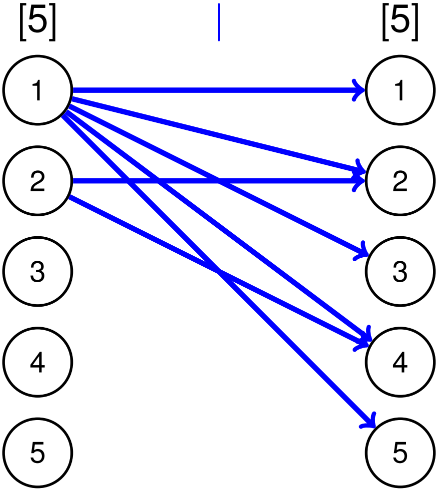

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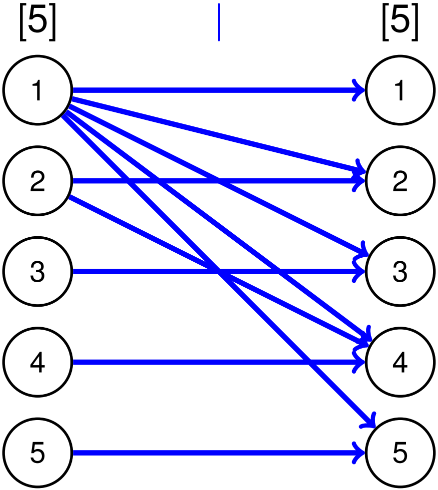

$\{(1,1),(1,2),(1,3),(1,4),(1,5),(2,2),(2,4),(3,3),(4,4),(5,5)\}$

Functions $f : A \to B$ $\quad (a, b) \in f$ iff $f(a) = b$

Def: f is a **function** from A to B ($f : A \to B$) iff $f \subseteq A \times B$, and

f is **defined** on domain A: $\forall a \in A\, \exists b \in B\, (a, b) \in f$, and

f is **single valued**: $\forall (a, b), (a', b') \in f\ (a = a' \ \to\ b = b')$.

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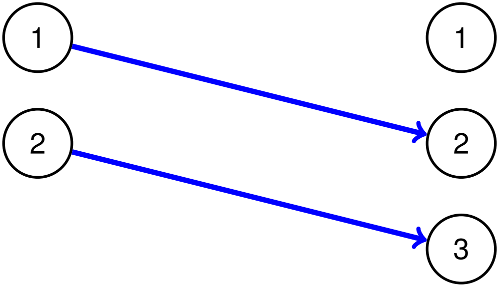
$f : [2] \to [3]$

$f(x) = x + 1$

[2] [3]



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$\mathrm{id}_{[2]} : [2] \to [2]$

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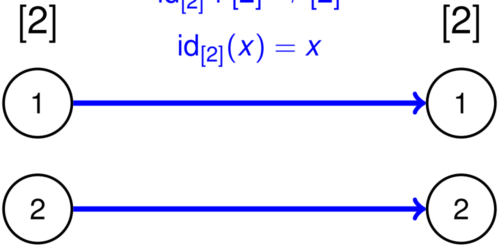

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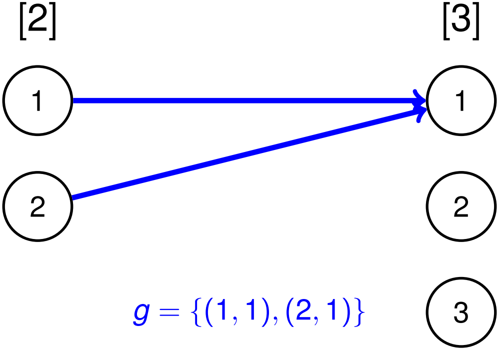

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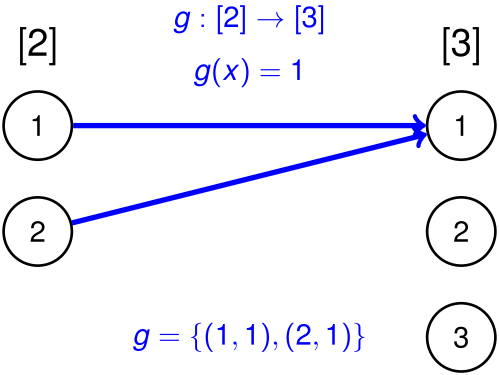
A: Yes



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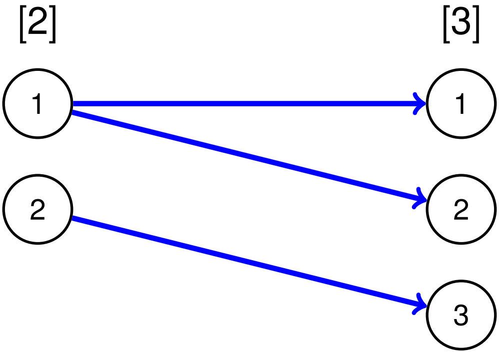

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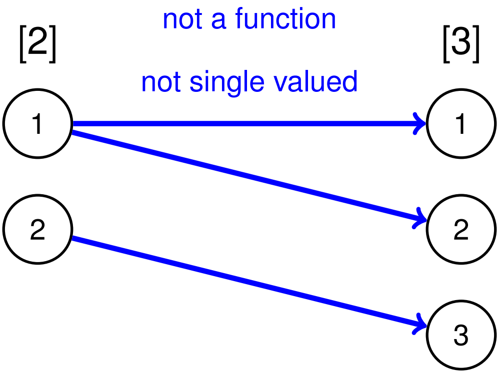
B: No



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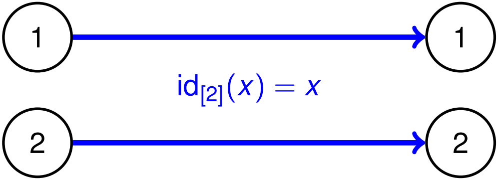

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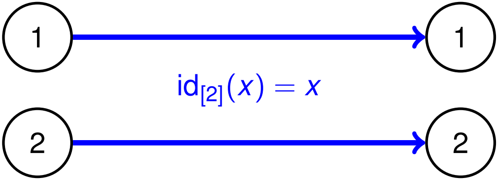

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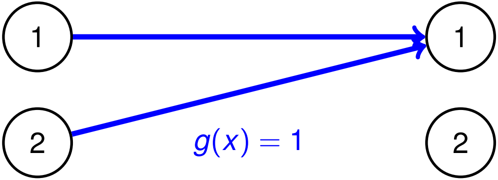

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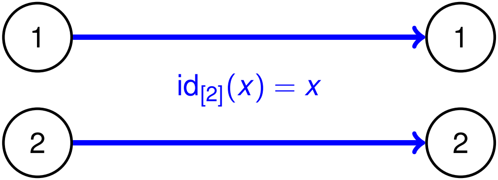

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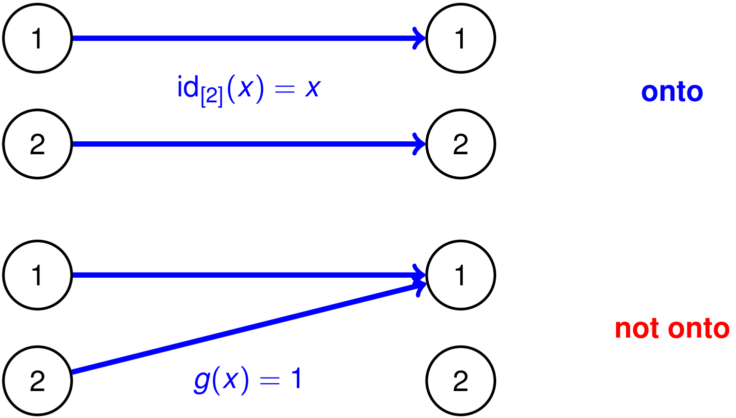

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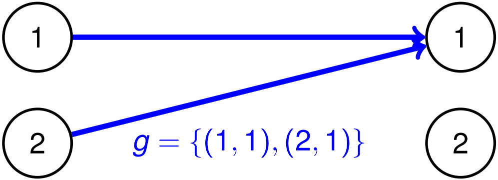

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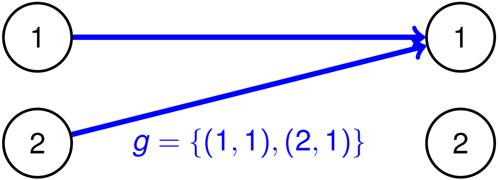

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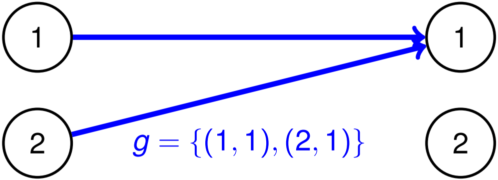

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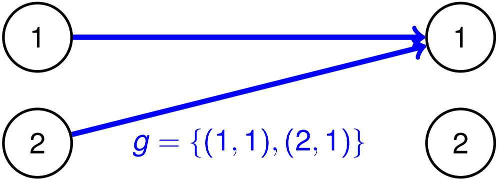

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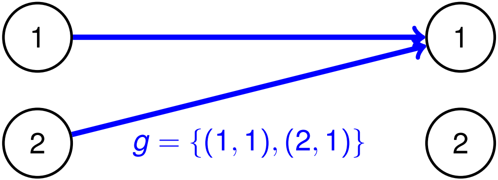

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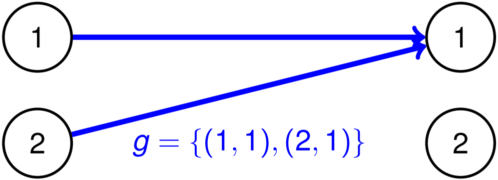

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For $f \subseteq A \times B$, we can tell if f is **single valued** and if it is **1:1**.

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For $f \subseteq A \times B$, we can tell if f is **single valued** and if it is **1:1**.

To tell if $f : A \to B$, i.e., is f a **function**, we must know A.

Domain, Range, and Co-Domain

For $f : A \to B$, it's **domain** and **range** are well defined.

Def. The **domain** of f: $\mathrm{dom}(f) \stackrel{\text{def}}{=} \{a \mid \exists b\, (a, b) \in f\} = A$

Def. The **range** of f: $\mathrm{rng}(f) \stackrel{\text{def}}{=} \{b \mid \exists a\, (a, b) \in f\}$

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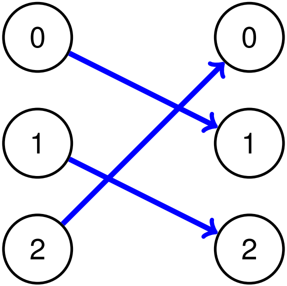

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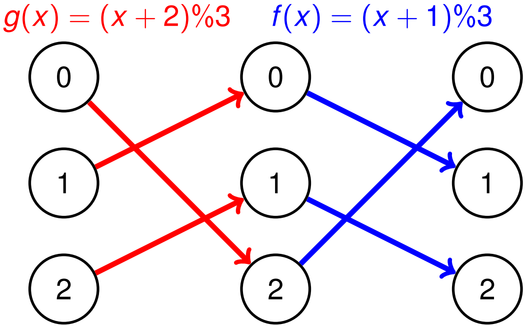

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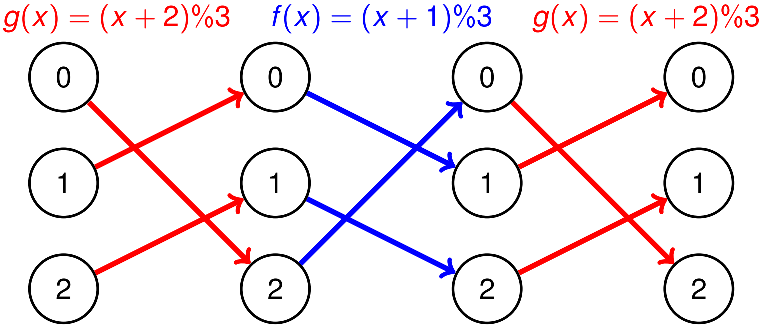

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Check on your own. We'll talk about this more next week. $\square$