CS250: Discrete Math for Computer Science

L22: Inductive Definitions and Structural Induction

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- Modern programming langugages allow recursive function definitions on recursively defined datatypes (Python – hw4)
- Main examples today: logical formulas, truth

Terms in PredCalc

Def: Let Σ be a PredCalc vocabulary. A **term** $t \in \textbf{term}(\Sigma)$ is a string of symbols that every world $W \in \mathrm{World}[\Sigma]$ must interpret as an element $t^W \in |W|$. Terms are defined recursively as follows:

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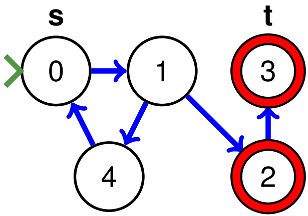
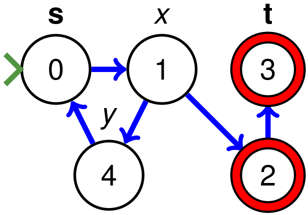
$$\textbf{term}(\Sigma_{\text{\#thy}}) \;=\; \{0, 1, x, \ldots, \ldots, x \cdot y, \ldots (x+1) \cdot (y+0), \ldots\}$$



G_1



$G_1[1/x\ 4/y]$

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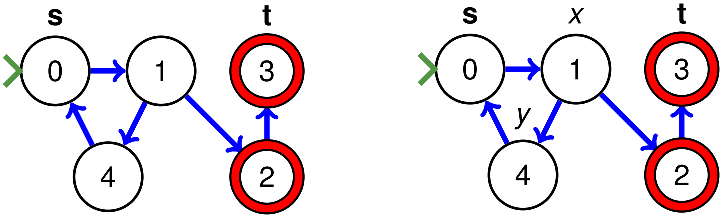

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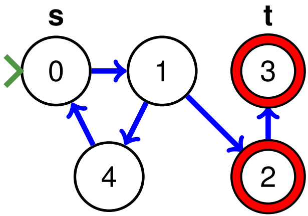

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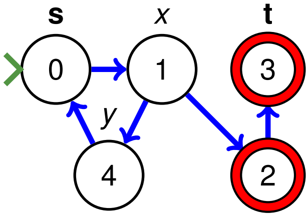

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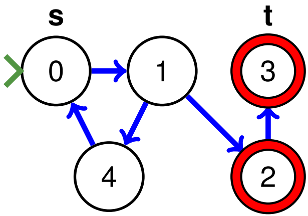

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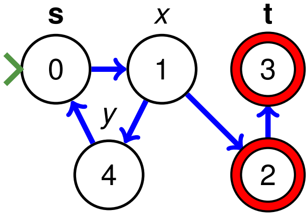

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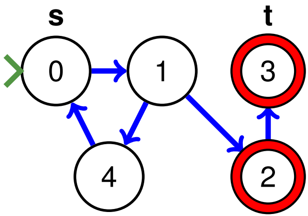

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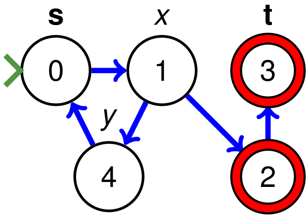

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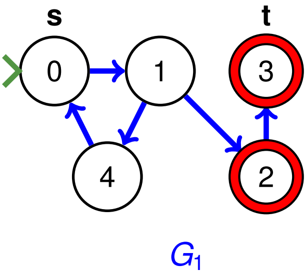

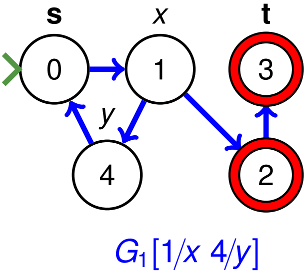

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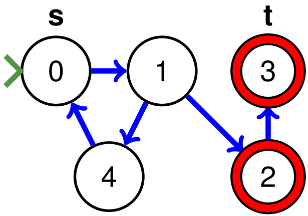
iClicker 22.1 What is $y^{G_1[1/x\ 4/y]}$

A: 0 B: 1 C: 3 D: 4



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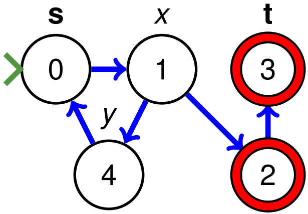

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$f^W : |W|^r \to |W|$, so $(f(t_1, \ldots, t_r))^W = f^W(t_1^W, \ldots, t_r^W) \in |W|$. $\square$

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$$G \models \sim\alpha \quad \text{iff} \quad G \not\models \alpha \qquad \text{PropCalc}$$

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G	$\models$	$t_1 = t_2$	iff	$t_1^G = t_2^G$	
G	$\models$	$P(t_1, \ldots, t_a)$	iff	$(t_1^G, \ldots, t_a^G) \in P^G$	$P^a \in \Sigma$
G	$\models$	$\sim\alpha$	iff	$G \not\models \alpha$	PropCalc
G	$\models$	$\alpha \wedge \beta$	iff	$G \models \alpha$ **and** $G \models \beta$	PropCalc

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$$
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G & \models & \alpha \wedge \beta & \text{iff} & G \models \alpha \quad \textbf{and} \quad G \models \beta & \text{PropCalc} \\
G & \models & \alpha \vee \beta & \text{iff} & G \models \alpha \quad \textbf{or} \quad G \models \beta & \text{PropCalc}
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G	$\models$	$\exists x(\alpha)$	iff	**exists** $a \in \lvert G \rvert \quad G[a/x] \models \alpha$	

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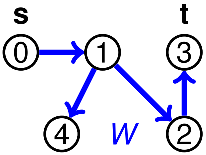
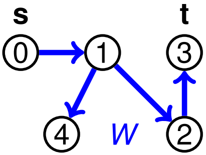
$W \models \exists x\, \varphi$	**D** chooses	$a \in \lvert W \rvert$ continue on:	$W\, a/x \models \varphi$

Truth Game Example



Does $W \models \forall x\ (x = s \vee \exists y\ E(y, x))$?

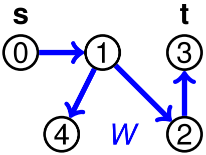
Truth Game Example



Does $W \models \forall x\ (x = s \vee \exists y\ E(y, x))$?

G moves, chooses $x = 4$

Truth Game Example



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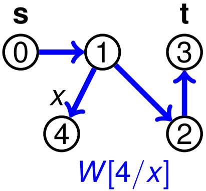

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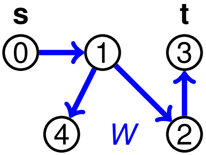

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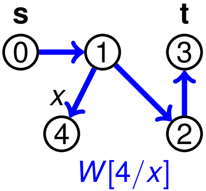

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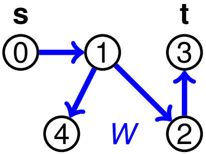

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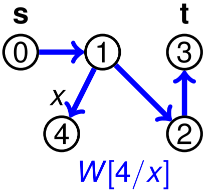

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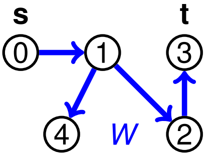

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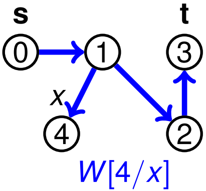

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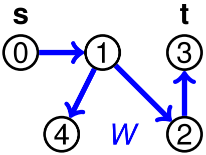
D moves, chooses $y = 1$

Truth Game Example



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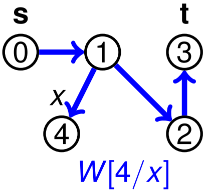

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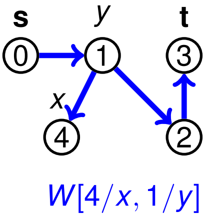

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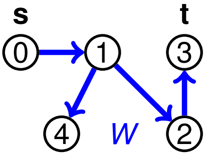
Truth Game Example



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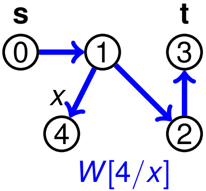

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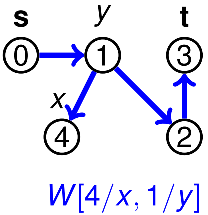

Does $W[4/x, 1/y] \models E(y, x)$?

Yes, **D** wins!

Tarski's Def. of Truth and Truth Game are Equivalent

Thm. For any Σ, $\varphi \in \mathrm{PredCalc}\Sigma$, in NNF, $\mathcal{W} \in \mathrm{World}[\Sigma]$,

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Details in hw4 $\square$