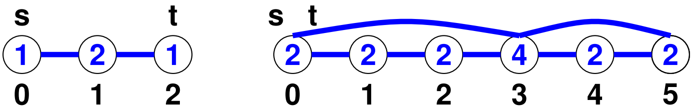
CS250: Discrete Math for Computer Science

L20: Complete Induction and Proof of Euler's Characterization of Eulerian-Walks

Last time: Eulerian Graphs



Def. An **Eulerian walk** in a graph G is a walk from s to t that traverses every edge exactly once and every vertex at least once.

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Thm. [Euler] G has an Eulerian walk from s to t iff $\mathrm{EC}(G)$.

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Claim G has an Eulerian walk $\rightarrow$ $\mathrm{EC}(G)$.

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Proof: For all vertices, v, besides s and t, the walk must leave v the same number of times that it enters v. Thus, deg(v) is even.

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Since an Eulerian walk visits every vertex, G is connected. $\square$

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We will prove the Claim by induction: $\mathbf{N} \models \forall x \ \alpha(x)$, where

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base case: $\alpha(0)$: Let G be an arbitrary graph with 0 edges and $\mathrm{EC}(G)$. Since G is connected and has no edges it must consist of a single vertex, $s = t$. Thus, the empty walk is an Eulerian walk from s to t. ✓

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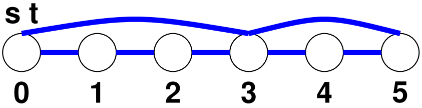
Let G be an arbitrary graph with $x_0 + 1$ edges and $\mathrm{EC}(G)$.



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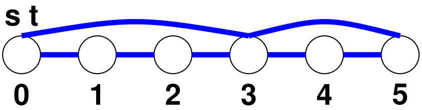
Using $\alpha(x_0)$, **construct an Eulerian walk** through G.



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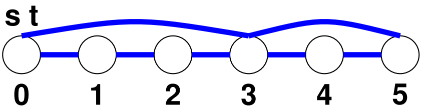

An **inductive proof** is a **recursive algorithm**: using calls to $\alpha(x_0)$, constructing Eulerian walks on graphs with at most x_0 edges, we must **construct an Eulerian walk** through G.

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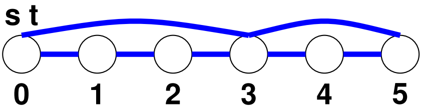

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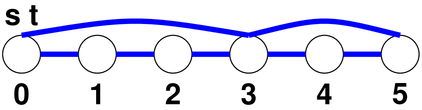

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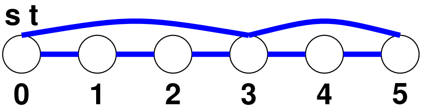

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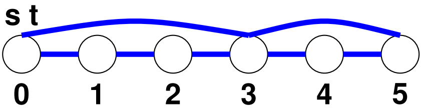

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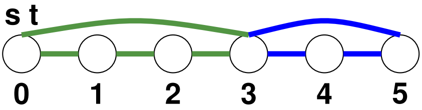

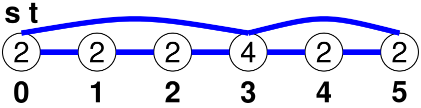

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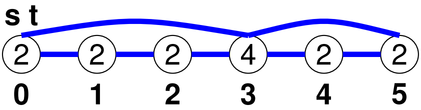

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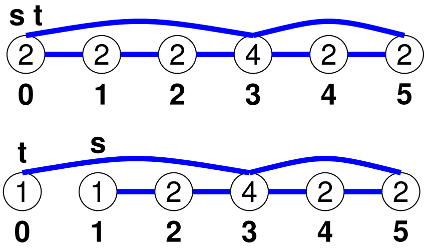

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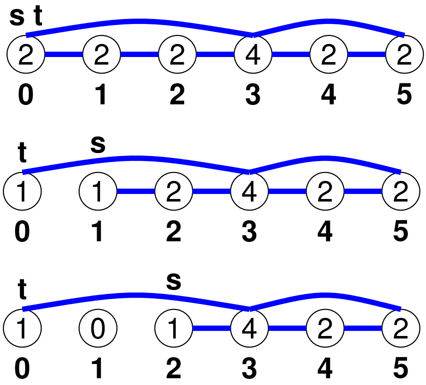

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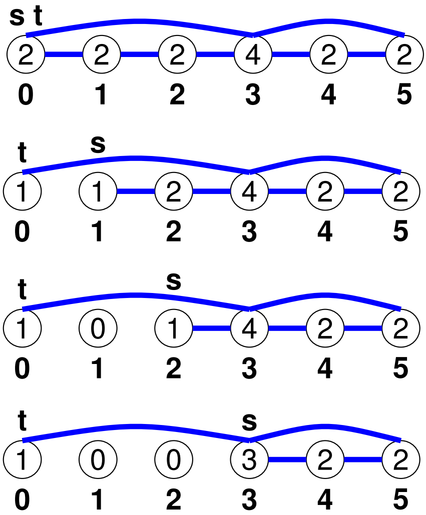

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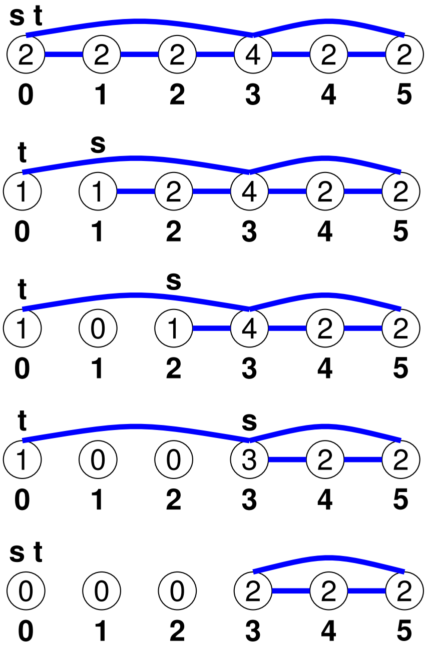

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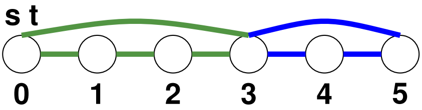

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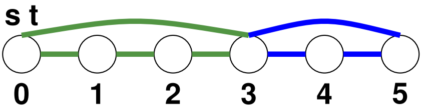

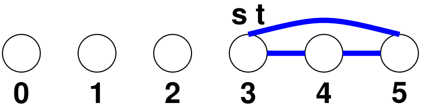

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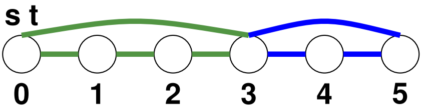

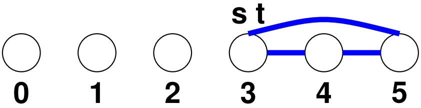

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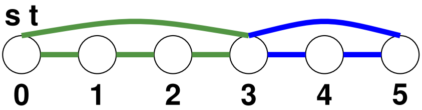

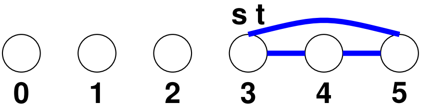

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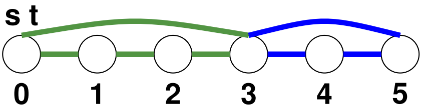

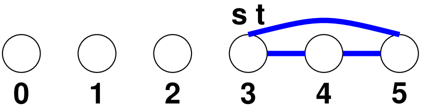

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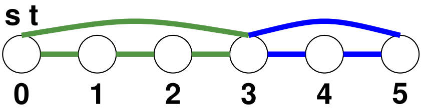

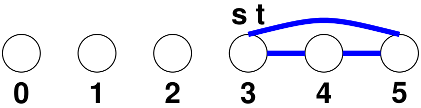

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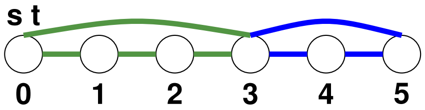

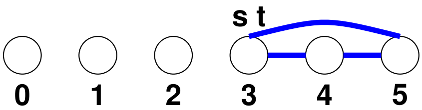

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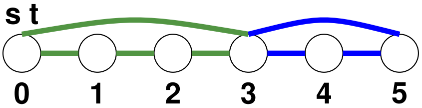
By **indHyp** each C has an Eulerian walk. (**recursive call**)

The union of these Eulerian walks is an Eulerian walk for G .



✓

iClicker 20.1 Did we need the assumption EC(G) to know that in G, a **bb** walk from s always ends at t?

A: Yes **B: No**

iClicker 20.2 In the proof of Euler's characterization of Eulerian Graphs, why did we need α to say $\|E^G\| \leq x_0$ as opposed to $\|E^G\| = x_0$?

A: We didn't, $\|E^G\| = x_0$ would have been enough.

B: We wanted to apply the inductive hypothesis to graphs with fewer than $x_0 + 1$ vertices. We didn't have control of exactly how many edges were removed in going from G to the smaller graphs.

In **complete induction**, we strengthen the inductive hypothesis from $\alpha(x_0)$ to $\forall z \leq x_0 \, (\alpha(z))$.

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In order to prove $\forall x\, \alpha(x)$ by complete induction, we just prove $\forall x\, \alpha_c(x)$ by normal induction, where,

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This little trick allows us to use the stronger – complete – inductive hypothesis whenever we want to!

In the proof of Euler's characterization of Eulerian graphs, we used complete induction:

$$\alpha(x_0) \equiv \forall G\,(|E^G| \leq x_0 \wedge \mathrm{EC}(G) \ \rightarrow\ G \text{ has an Eulerian walk })$$

iClicker 20.3 What is the difference between complete induction and ordinary induction?

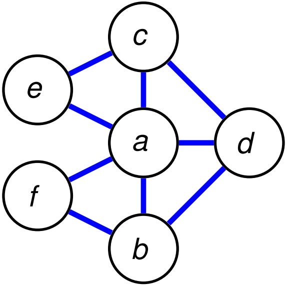
A: Complete induction just consists of certain instances of ordinary induction, in which $\alpha(x)$ is of the form $\forall z \leq x\,(\psi(z))$.

B: Complete induction is stronger than ordinary induction in the sense that there are worlds that are total orderings satisfying all of ordinary induction, but not complete induction.

$v = |V| \quad e = |E| \quad f = \#$ of faces

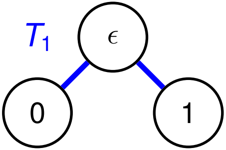
connected plane graphs

G_1

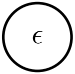


G	v	e	f	$v - e + f$

T_1



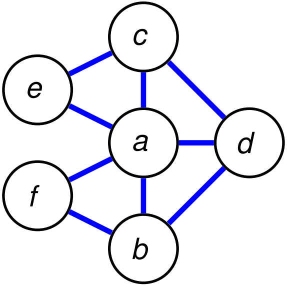
T_0



$v = |V| \quad e = |E| \quad f = \#$ of faces

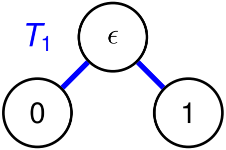
connected plane graphs

G_1

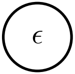


G	v	e	f	$v - e + f$
G_1	6	9	5	2

T_1



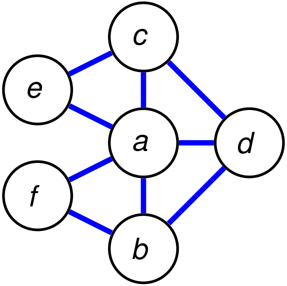
T_0



$v = |V| \quad e = |E| \quad f = \#$ of faces

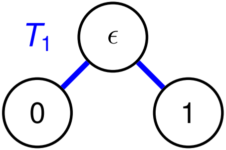
connected plane graphs

G_1

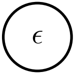


G	v	e	f	$v - e + f$
G_1	6	9	5	2
T_0	1	0	1	2

T_1



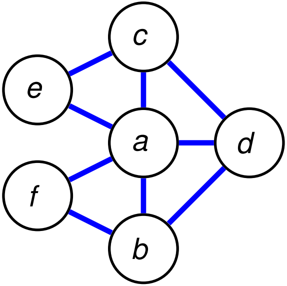
T_0



$v = |V| \quad e = |E| \quad f = \#$ of faces

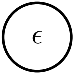
connected plane graphs

G_1

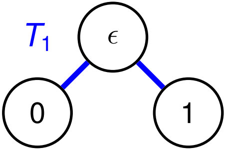


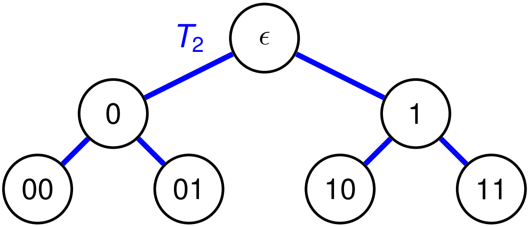
G	v	e	f	$v - e + f$
G_1	6	9	5	2
T_0	1	0	1	2
T_1	3	2	1	2

T_0

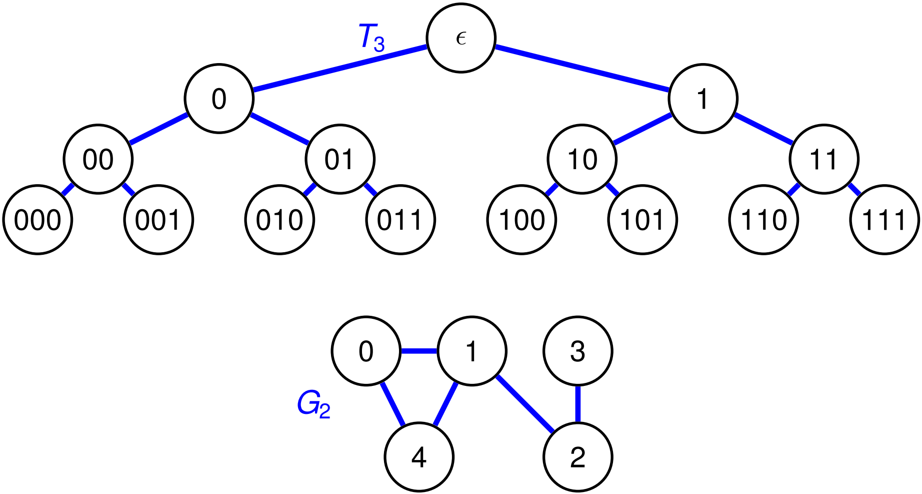


T_1





T_2
ϵ
0
1
00
01
10
11


T_3
ϵ
0
1
00
01
10
11
000
001
010
011
100
101
110
111
G_2
0
1
3
4
2

Thm: Euler's Formula [1750] Let G be an undirected, connected graph, drawn in the plane. Then $v - e + f = 2$.

We will prove this by induction next time.