CS250: Discrete Math for Computer Science

L10: Natural Deduction for PredCalc

Tarski's Recursive Definition of Truth

For every $G \in \mathrm{World}[\Sigma]$ and $t \in \textbf{term}(\Sigma)$ $\qquad t^G \in |G|$

G	$\models$	$t_1 = t_2$	iff	$t_1^G = t_2^G$	
G	$\models$	$P(t_1, \ldots, t_a)$	iff	$(t_1^G, \ldots, t_a^G) \in P^G$	$P^a \in \Sigma$
G	$\models$	$\sim\alpha$	iff	$G \not\models \alpha$	PropCalc
G	$\models$	$\alpha \wedge \beta$	iff	$G \models \alpha$ **and** $G \models \beta$	PropCalc
G	$\models$	$\alpha \vee \beta$	iff	$G \models \alpha$ **or** $G \models \beta$	PropCalc
G	$\models$	$\forall x(\alpha)$	iff	**for all** $a \in \|G\|$ $\quad G[a/x] \models \alpha$	
G	$\models$	$\exists x(\alpha)$	iff	**exists** $a \in \|G\|$ $\quad G[a/x] \models \alpha$	

Remaining Natural Deduction Proof Rules

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

Remaining Natural Deduction Proof Rules

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$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
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$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

1	$x = x$	$=$-i

Remaining Natural Deduction Proof Rules

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

1	$x = x$	$=$-i

1	$x = y$	
2	$R(x)$	
3	$R(y)$	$=$-e, 1, 2

Remaining Natural Deduction Proof Rules

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

Prop: By Tarski's Def of Truth: $=$-i and $=$-e are sound.

1	$x = x$	$=$-i

1	$x = y$	
2	$R(x)$	
3	$R(y)$	$=$-e, 1, 2

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

	introduction	elimination
$=$	$\overline{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

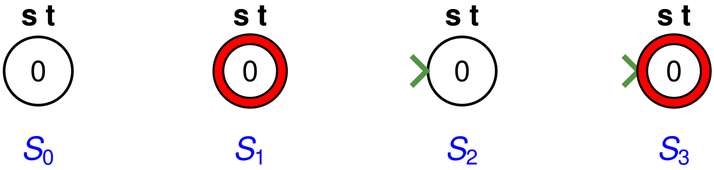
1	$\forall x\,(A(x) \to R(x))$	
2	$A(z) \to R(z)$	$\forall$-e, 1

	introduction	elimination
$=$	$\overline{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x \; \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x \; \varphi}$	

Prop: By Tarski's Def of Truth: $\forall$-e is sound.

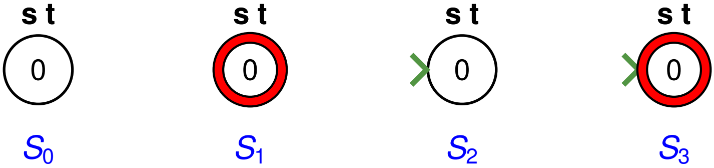
1	$\forall x \, (A(x) \to R(x))$	
2	$A(z) \to R(z)$	$\forall$-e, 1

R10 Quiz Solutions



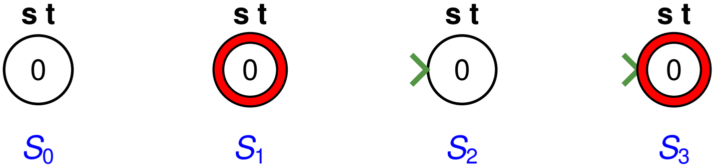
1	$\forall x\,(A(x) \to R(x))$	
2	$A(s)$	
3	$A(s) \to R(s)$	$\forall$-e, 1
4	$R(s)$	$\to$-e, 2, 3

R10 Quiz Solutions



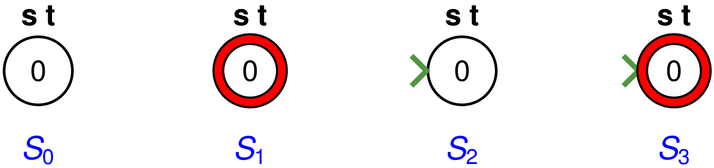
1	$\forall x\,(A(x) \to R(x))$	
2	$R(s)$	
3	$A(s) \to R(s)$	$\forall$-e, 1
4	$A(s)$	**No:** S_1 is counterexample

R10 Quiz Solutions



1	$\forall x\,(A(x) \to R(x))$	
2	$\sim A(s)$	
3	$A(s) \to R(s)$	$\forall$-e, 1
4	$\sim R(s)$	**No:** S_1 is counterexample

R10 Quiz Solutions



1	$\forall x\,(A(x) \rightarrow R(x))$	
2	$\sim R(s)$	
3	$A(s) \rightarrow R(s)$	$\forall$-e, 1
4	$\sim A(s)$	$\rightarrow$-e, 2, 3

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

	introduction	elimination
$=$	$\overline{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

1	$\forall x\,(x + 0 = x)$	
2	$\exists y\,\forall x\ (x + y = x)$	$\exists$-i, 1

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

Prop: By Tarski's Def of Truth: $\exists$-i is sound.

1	$\forall x\,(x + 0 = x)$	
2	$\exists y\,\forall x\,(x + y = x)$	$\exists$-i, 1

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

Prop: By Tarski's Def of Truth: $\exists$-i is sound.

Proof.

Assume $G \models \varphi[t/x]$.

$\square$

1	$\forall x\,(x + 0 = x)$	
2	$\exists y\,\forall x\,(x + y = x)$	$\exists$-i, 1

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

Prop: By Tarski's Def of Truth: $\exists$-i is sound.

Proof.

Assume $G \models \varphi[t/x]$. $\quad a \stackrel{\text{def}}{=} t^G$

$\square$

$$
\begin{array}{l|ll}
1 & \forall x\,(x + 0 = x) & \\
\hline
2 & \exists y\,\forall x\,(x + y = x) & \exists\text{-i, } 1
\end{array}
$$

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

Prop: By Tarski's Def of Truth: $\exists$-i is sound.

Proof.

Assume $G \models \varphi[t/x]$. $\quad a \stackrel{\text{def}}{=} t^G \quad$ Thus, $G[a/x] \models \varphi$

$\square$

$$\begin{array}{r|ll} 1 & \forall x\,(x + 0 = x) & \\ \hline 2 & \exists y\,\forall x\,(x + y = x) & \exists\text{-i, } 1 \end{array}$$

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

Prop: By Tarski's Def of Truth: $\exists$-i is sound.

Proof.
Assume $G \models \varphi[t/x]$. $\quad a \stackrel{\text{def}}{=} t^G \quad$ Thus, $G[a/x] \models \varphi$

Thus, by Tarski's Def of Truth, $\quad G \models \exists x\ \varphi$ $\square$

1	$\forall x\,(x + 0 = x)$	
2	$\exists y\,\forall x\,(x + y = x)$	$\exists$-i, 1

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

$$\vdash\ \forall x\ \varphi\ \rightarrow\ \exists x\ \varphi$$

	introduction	elimination
$=$	$\overline{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

$$\vdash\ \forall x\ \varphi\ \rightarrow\ \exists x\ \varphi$$

1	$\quad \forall x\ \varphi$	
2	$\quad \varphi$	$\forall$-e, 1
3	$\quad \exists x\ \varphi$	$\exists$-i, 2
4	$\forall x\ \varphi\ \rightarrow\ \exists x\ \varphi$	$\rightarrow$-i, 1–3

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

$$\vdash\ \exists y\ f(x) = y$$

	introduction	elimination
$=$	$\dfrac{}{t = t}$	$\dfrac{t_1 = t_2 \quad \varphi[t_1/x]}{\varphi[t_2/x]}$
$\forall$		$\dfrac{\forall x\ \varphi}{\varphi[t/x]}$
$\exists$	$\dfrac{\varphi[t/x]}{\exists x\ \varphi}$	

$$\vdash\ \exists y\ f(x) = y$$

1	$f(x) = f(x)$	$=$-i
2	$\exists y\ f(x) = y$	$\exists$-i, 2

Solution to Disc. 3

$$\vdash \ \sim(p \vee \sim p)$$

1	$\sim(p \vee \sim p)$	
2	p	
3	$p \vee \sim p$	$\vee$-i, 2
4	**F**	**F**-i, 1, 3
5	$\sim p$	**F**-e, 2–4
6	$p \vee \sim p$	$\vee$-i, 5
7	**F**	**F**-i, 1, 6
8	$(p \vee \sim p)$	**F**-e, 1–7

Hw1 Answers

1a. $\sim d \equiv$ "The election is not decided."

1b. $d \wedge \sim c$ "The election is decided but the votes have not been counted."

1c. $c \rightarrow d$ "The election is decided if the votes have been counted."

1d. $\sim d \rightarrow \sim c$ "If the election is not decided then the votes have not been counted."

Hw1 Answers

2a.	$a \wedge \sim f$	“You get an A in the course, without getting a perfect score on the final.”
2b.	$w \vee f \rightarrow a$	“To get an A in the course, it is sufficient to do all the assigned work, or, to get a perfect score on the final.”
2c.	$\sim f \rightarrow \sim w \rightarrow \sim a$ $\equiv$ $\sim f \wedge \sim w \rightarrow \sim a$	“If you don’t do all the assigned work then you won’t get an A in the course unless you get a perfect score on the final.”
2d.	$\sim a \rightarrow \sim w$	“If you don’t get an A in the course, then you didn’t do all the assigned work.”

Hw1: Knights or Knaves?

$T_1 \stackrel{\text{def}}{=}$ A : "I'm only Kt or only Kv"

$T_2 \stackrel{\text{def}}{=}$ B : "Exactly 1 Kv"

$T_3 \stackrel{\text{def}}{=}$ A : "Exactly 1 of A, B is Kt"

$S_1 \equiv$ $T_1 \leftrightarrow A$ is Kt

$S_2 \equiv$ $T_2 \leftrightarrow B$ is Kt

$S_3 \equiv$ $T_3 \leftrightarrow C$ is Kt

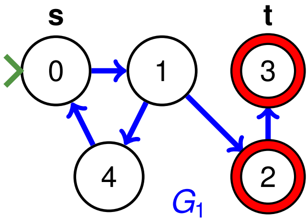
W	A	B	C	T_1	T_2	T_3	S_1	S_2	S_3
W_7	1	1	1	0	0	0	0	0	0
W_6	1	1	0	0	1	0	0	1	1
W_5	1	0	1	0	1	1	0	0	1
W_4	1	0	0	1	0	1	1	1	0
W_3	0	1	1	1	1	1	0	1	1
W_2	0	1	0	0	0	1	1	0	0
W_1	0	0	1	0	0	0	1	1	0
W_0	0	0	0	0	0	0	1	1	1

W_0 is the unique world satisfying $S_1 \wedge S_2 \wedge S_3$. Thus, A, B and C are **all knaves**.

Hw1: Convert PropCalc Formula to CNF

$$
\begin{aligned}
&{\sim}((\sim p \wedge q \wedge r) \vee\ {\sim}(p \rightarrow (q \rightarrow r))) \quad \equiv \\
&\quad \equiv \quad {\sim}(\sim p \wedge q \wedge r)\ \wedge\ (p \rightarrow (q \rightarrow r)) \\
&\quad \equiv \quad (p \vee\ {\sim} q \vee\ {\sim} r)\ \wedge\ ({\sim} p \vee\ {\sim} q\ \vee\ r)
\end{aligned}
$$

R9 Quiz Answers



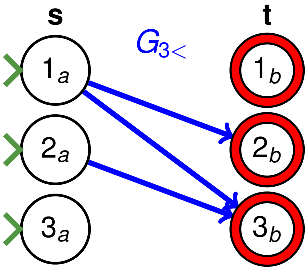
1. $G_1 \models \forall x \exists y\, (R(x) \rightarrow E(y,x))$ **True**
2. $G_1 \models \exists x \forall y\, (R(x) \wedge \sim E(y,x))$ **False**
3. $G_1 \models \forall x \exists y\, (R(x) \rightarrow E(x,y))$ **False** counterexample: $x = 3$
4. $G_1 \models \exists x \forall y\, (R(x) \wedge \sim E(x,y))$ **True** witness: $x = 3$

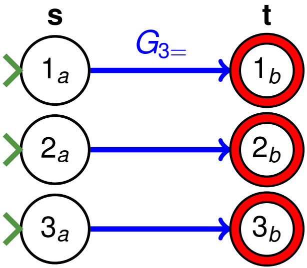
R9 Quiz Answers

Convert each of the following PredCalc formulas to NNF.

5. $\sim\forall x \exists y\,(R(x) \rightarrow E(y,x)) \quad\equiv\quad \exists x \forall y\,(R(x) \wedge \sim E(x,y))$
6. $\sim\exists x \forall y\,(R(x) \wedge \sim E(y,x)) \quad\equiv\quad \forall x \exists y\,(\sim R(x) \;\vee\; E(y,x))$
7. $\sim\forall x \exists y\,(R(x) \rightarrow E(x,y)) \quad\equiv\quad \exists x \forall y\,(R(x) \wedge \sim E(x,y))$
8. $\sim\exists x \forall y\,(R(x) \wedge \sim E(x,y)) \quad\equiv\quad \forall x \exists y\,(\sim R(x) \;\vee\; E(x,y))$

R9 Quiz Answers





9. T/F: $G_{3<} \models \forall xy(E(x,y) \to A(x) \wedge R(y))$ **True**

10. T/F: $G_{3=} \models \forall xy(E(x,y) \to A(x) \wedge R(y))$ **True**

11. T/F: $G_{3<} \models \forall x \exists y(A(x) \to E(x,y))$ **False**

12. T/F: $G_{3=} \models \forall x \exists y(A(x) \to E(x,y))$ **True**

13. T/F: $G_{3<} \models \forall xyz(E(x,y) \wedge E(x,z) \to y = z)$ **False**

14. T/F: $G_{3=} \models \forall xyz(E(x,y) \wedge E(x,z) \to y = z)$ **True**