Computer Science 791DD, Learning to See

http://www.cs.umass.edu/∼elm/learning2see/

Assignment 4

1. Suppose a continuous random variable has a distribution whose support is limited to the interval $[a,b]$. Answer the following questions (5 points each)

   (a) Give an upper bound, in terms of a and b, on the differential entropy of this random variable.

   (b) What is the tightest lower bound?

   (c) Now suppose you are told that the maximum height of the probability density is d, while the support is still limited to $[a,b]$. Can you put a tighter lower bound on the differential entropy? What is it?

   (d) If you remove the restriction on the support, but the density is still limited in height to d, what is the tightest upper bound on the differential entropy? The tightest lower bound?

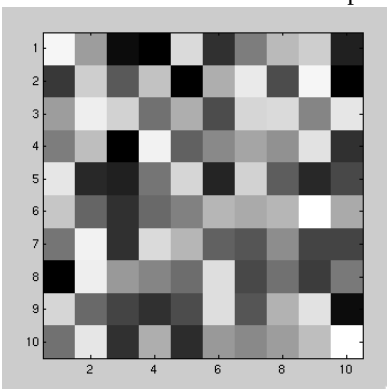
2. Matlab functions. (10 points each)

   (a) Write a Matlab function discreteEntropy.m that computes the entropy of a discrete probability distribution, i.e. a probability mass function. Assume that the input is a vector of probabilities of events. Make sure the function can handle zero probabilities. Return the entropy in bits. Show the output of a function for a particular example.

   (b) Use the function you just wrote to write another function, mutualInformation.m, which computes the mutual information between two random variables. Assume that the joint probability distribution, in the form of a two-dimensional matrix, is given as the input. Hint: the body of this function can easily be written in a single line in Matlab.

   (c) Finally, write a function to compute the KL-divergence between two discrete distributions. It should take the two distributions as inputs (as vectors). If the true KL-divergence is infinity, then this is what your function should return.

3. The following figure shows the joint distribution of two random variables X and Y, each of which takes on 10 different values. The probability of each joint event, $P(X = x, Y = y)$ is represented by its brightness, where black is 0 and white is the highest possible value. Are X and Y independent? Give an airtight argument for your answer. Remember that this is a probability distribution, not a *sample* from a probability distribution. (5 points)



4. George Bush has an algorithm that only runs on "grayscale" images, meaning images in which each pixel is an integer value in the interval $[0,255]$ inclusive, representing the brightness of that pixel. That is, each pixel needs to be an 8-bit value. The algorithm cannot use color information. George has an image to which he wants to apply his algorithm, but it is a color image, with 24-bits per pixel. The first 8 bits of a pixel represent red, the next 8 green, and the next 8 blue. He uses a matlab command (rgb2hsv) to convert the red-green-blue image to hue-saturation-value, in which the last channel, "value", can be interpreted as the brightness of the image. He uses this last channel as his gray-scale image.
   Unfortunately, after converting the image to grayscale, many of the brightness values in the image are exactly the same. In particular, the distribution of brightness values is as follows:

$$P(0) \quad \frac{1}{2} \tag{1}$$

$$P(63) \quad \frac{1}{16} \tag{2}$$

$$P(127) \quad \frac{1}{16} \tag{3}$$

$$P(253) \quad \frac{1}{8} \tag{4}$$

$$P(254) \quad \frac{1}{8} \tag{5}$$

$$P(255) \quad \frac{1}{8} \tag{6}$$

   (a) The number of "bits" of information in this image can be approximated by the number of pixels times the entropy of the distribution of pixel values. Calculate this number. (2 points)

   (b) George decides he wants his image to contain more "information", so he takes the image pixels whose values are 0 and changes them randomly to have values from 0-62. He changes the values in the other bins similarly. Will the entropy of the distribution of brightness values go up, go down, or stay the same? Why? (2 points).

   (c) George claims he has increased the amount of "information" in the image. What do you think of this argument? (5 points).

   (d) George's friend Ralph notices that Matlab's rgb2hsv code uses the command $v = \max(r,g,b)$ to calculate the brightness of an rgb pixel. He argues that the function is throwing away information about the pixel brightness since the pixels $(237,0,0)$, $(237,25,42)$, and $(237,0,1)$ would all get mapped to the same brightness value. He proposed to use the formula $v = \max(r,g,b) + \min(r,g,b)/256$ to produce a pixel's brightness value. Of course, truncating such a value, he points out, would produce the same result as the original brightness function. Instead, he does histogram equalization on the values *before* truncating them. His procedure results in an image whose values are distributed uniformly across brightness. He argues that his procedure does indeed preserve more information in the image. Is he deluding himself? Be as clear as you can be in your answer. (10 points)