Applied Information Theory 650

`http://www.cs.umass.edu/~elm/Teaching/650_F14/`

Assignment 1

1. *Computing entropy.* (2 points each) In this problem, you are to compute the entropy of various probability distributions, represented by sets of numbers in brackets. If there are K numbers in brackets, then there are K possible values of the corresponding random variable, whose probabilities are given by the numbers. In other words, a fair die would be written $[\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}]$. Compute the entropy of the following distributions (don't use a calculator unless the problem says you can):

- a $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]$
- b $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, 0, 0, 0]$
- c Compare answers **a** and **b**. Make a general statement about entropy calculations.
- d $[\frac{1}{4}, \frac{1}{4}, \frac{1.1}{4}, \frac{0.9}{4}]$ (you can use a calculator for this one)
- e Compare answers **a** and **d**. Can you offer a *conjecture* based upon these? (No need to prove it right now.)
- f $[\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}]$
- g $[\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}]$

2. A die is labeled from 1 to 6. Assuming it's fair, what is the entropy of the roll (you can leave a "log" in your answer)? The die is relabeled with the even numbers from 2 to 12. What is its entropy now? (2 points)

3. *Independence and entropy.* (A) Give the entropy, in bits, of four fair, 8-sided dice. (B). Suppose you drill a hole in each die, and tie them all together with a string. Will the entropy of the dice be higher or lower than the answer from part (A)? Why? (10 points).

4. *Coin Flips.* (From **Cover and Thomas, 2nd edition, Problem 2.1, part (a)**). (10 points). A fair coin is flipped until the first head occurs. Let X denote the number of flips required. Find the entropy $H(X)$ in bits. The following expressions may be useful:

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \qquad \sum_{n=0}^{\infty} nr^n = \frac{r}{(1-r)^2}$$

5. *Entropy of functions.* (From **Cover and Thomas, 2nd edition, Problem 2.2**). (5 points). Let X be a random variable taking on a finite number of values. What is the (general) inequality relationship of $H(X)$ and $H(Y)$ if

(a) $Y = 2^X$?

(b) $Y = cos(X)$?

6. *Minimum entropy.* (From **Cover and Thomas, 2nd edition, Problem 2.3**). (5 points). What is the minimum value of $H(p_1, ..., p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of n-dimensional probability vectors? Find all $\mathbf{p}$'s that achieve this minimum.

7. *Entropy of functions of a random variable.* (From **Cover and Thomas, 2nd edition, Problem 2.4**). (5 points). Let X be a discrete random variable. Show that the entropy of a function of X is less than or equal to the entropy of X by justifying the following steps:

$$\begin{aligned} H(X, g(X)) &\overset{\textbf{(a)}}{=} H(X) + H(g(X)|X) \\ &\overset{\textbf{(b)}}{=} H(X), \\ H(X, g(X)) &\overset{\textbf{(c)}}{=} H(g(X)) + H(X|g(X)) \\ &\overset{\textbf{(d)}}{\geq} H(g(X)). \end{aligned}$$

Thus, $H(g(X)) \leq H(X)$.

8. *Conditional mutual information vs. unconditional mutual information.* (From **Cover and Thomas, 2nd edition, Problem 2.6**). (5 points). Give examples of joint random variables X, Y, and Z such that

(a) $I(X; Y|Z) < I(X; Y)$.

(b) $I(X; Y|Z) > I(X; Y)$.

9. *Example of joint entropy.* (From **Cover and Thomas, 2nd edition, Problem 2.12, parts (a)-(e)**). (2 points each).

Let $p(x, y)$ be given by

		Y	
		0	1
X	0	1/3	1/3
	1	0	1/3

Find:

(a) $H(X), H(Y)$.

(b) $H(X|Y), H(Y|X)$.

(c) $H(X, Y)$.

(d) $H(Y) - H(Y|X)$.

(e) $I(X; Y)$.

10. *Relative entropy.* (From **Cover and Thomas, 2nd edition, Problem 2.37**). (10 points). Let X, Y, Z be three random variables with a joint probability mass function $p(x,y,z)$. The relative entropy between the joint distribution and the product of the marginals is the following:

$$D(p(x,y,z)||p(x)p(y)p(z)) = E\left[log\frac{p(x,y,z)}{p(x)p(y)p(z)}\right]$$

.

Expand this in terms of entropies. When is this quantity zero? Justify your answer in detail.

11. *Entropy and pairwise independence.* (From **Cover and Thomas, 2nd edition, Problem 2.39**). (10 points).

Let X, Y, Z be three binary Bernoulli($\frac{1}{2}$) random variables that are pairwise independent; that is, $I(X;Y) = I(X;Z) = I(Y;Z) = 0$.

(a) Under this constraint, what is the minimum value for $H(X,Y,Z)$?

(b) Give an example achieving this minimum.

For extra credit (up to 10 additional points), prove that your answer is one of the best possible answers.

12. *Mutual information of heads and tails.* (From **Cover and Thomas, 2nd edition, Problem 2.43**). (5 points).

(a) Consider a fair coin flip. What is the mutual information between the top and bottom sides of the coin?

(b) A six-sided fair die is rolled. What is the mutual information between the top side and the front face (the side most facing you)?