

# Monte Carlo Variational Inference

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Sample state  $z_1 \sim q$  and momentum  $\rho_1 \sim S$ .

Initialize estimator as  $\mathcal{L} \leftarrow -\log q(z_1)$ .

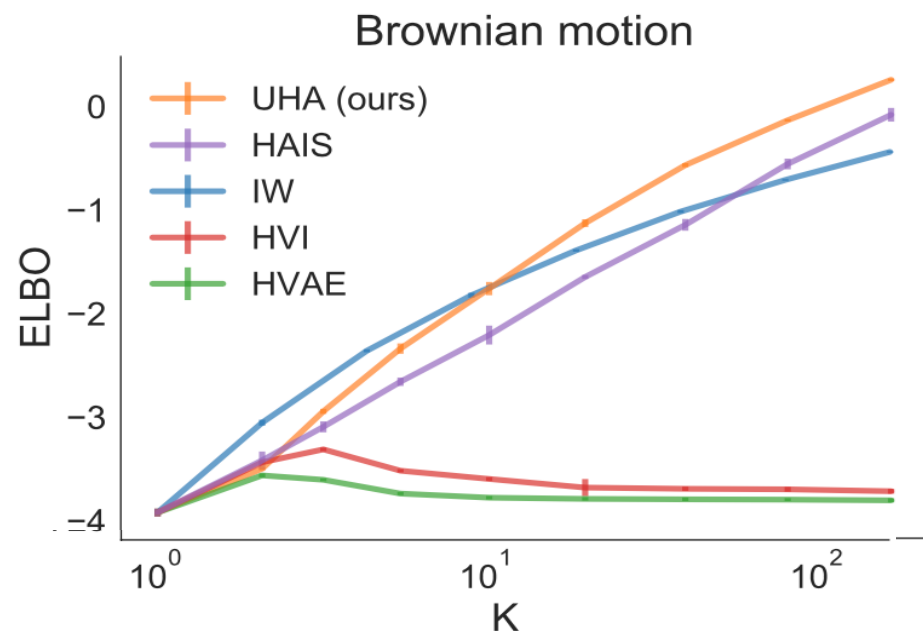
For  $m = 1, 2, \dots, K - 1$ :

    Sample new momentum  $\rho'$ .

$(z_{k+1}, \rho_{k+1}) \leftarrow \text{HMC with target } \pi_k \text{ and starting point } (z_k, \rho')$ .

    Update estimator as  $\mathcal{L} \leftarrow \mathcal{L} + \log S(\rho_{k+1}) - \log S(\rho')$

Update estimator as  $\mathcal{L} \leftarrow \mathcal{L} + \log p(z_K, x)$



# Calibration

Markov chain Monte Carlo

variational inference

sequential Monte Carlo

importance sampling

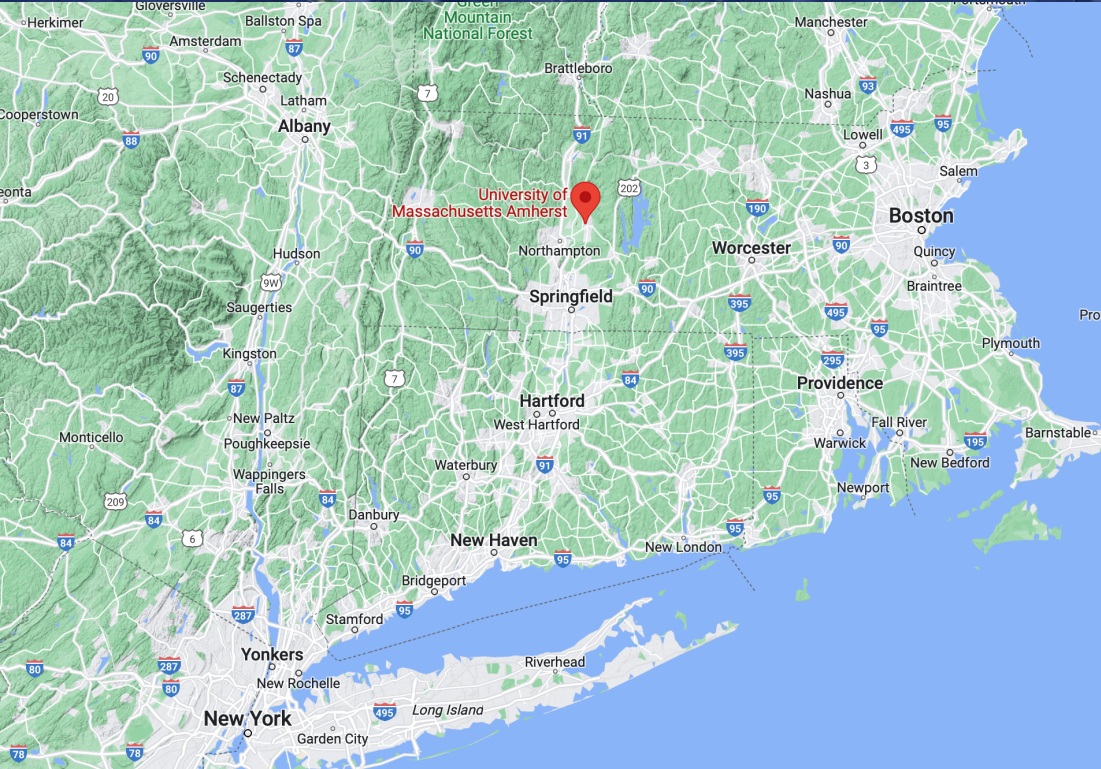
stratified sampling

KL-divergence

ELBO

Annealed importance sampling

Hamiltonian Monte Carlo

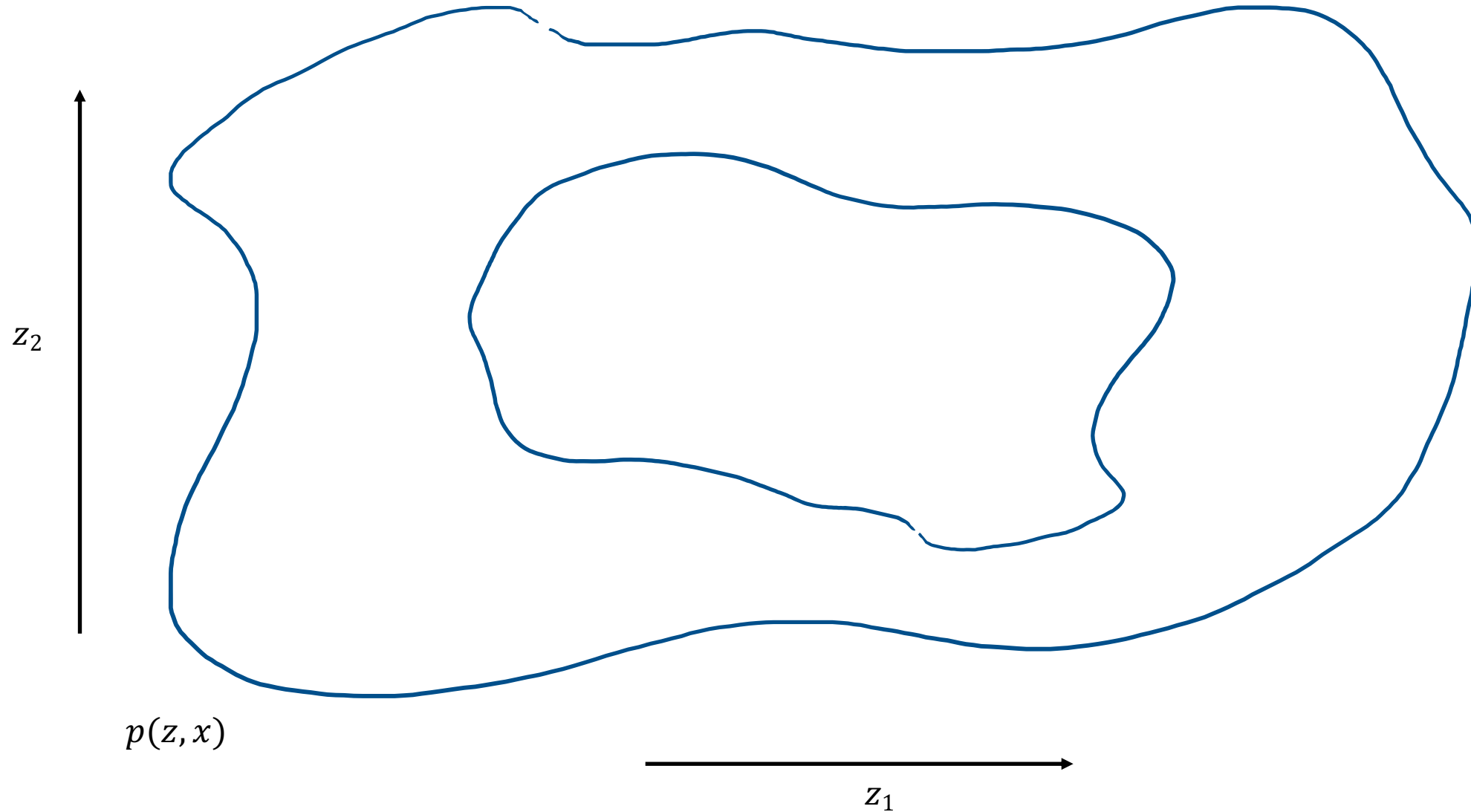


# Bayesian inference

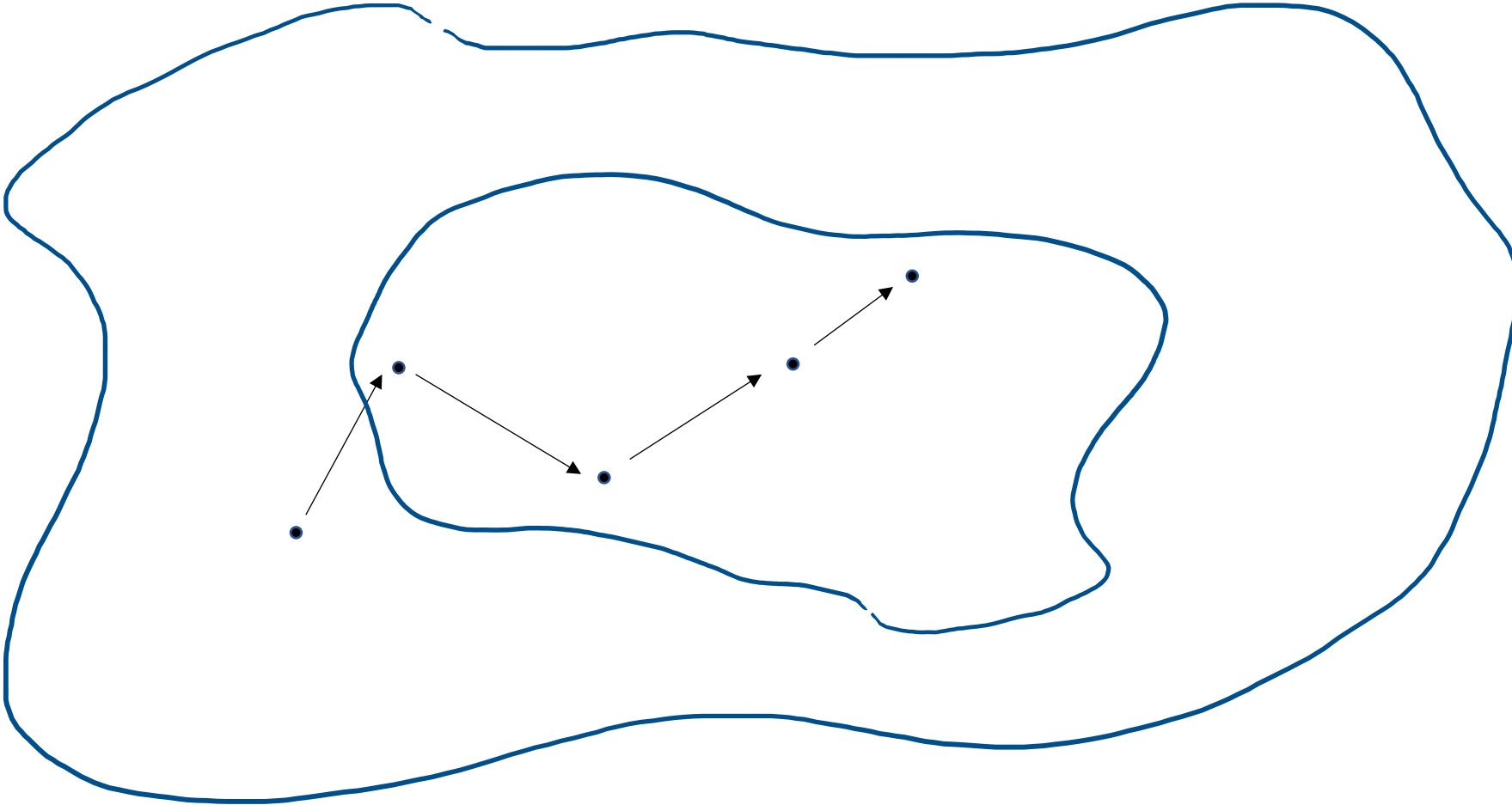
| domain            | latent $z$                                   | observed $x$                                     |
|-------------------|----------------------------------------------|--------------------------------------------------|
| epidemiology      | infections, transmission rate, recovery time | cases                                            |
| political science | candidate popularity, pollster biases        | polls                                            |
| astrophysics      | 14 parameters of the universe                | 2677 statistical measurements from galaxy survey |
| ecology           | # animals in different locations over time   | surveys                                          |
| phylogenetics     | evolutionary history                         | genomic data                                     |

1. Write down **prior**  $p(z)$ .
2. Write down **likelihood**  $p(x|z)$ .
3. Observe **data**  $x$ .
4. Use **posterior**  $p(z|x)$ .

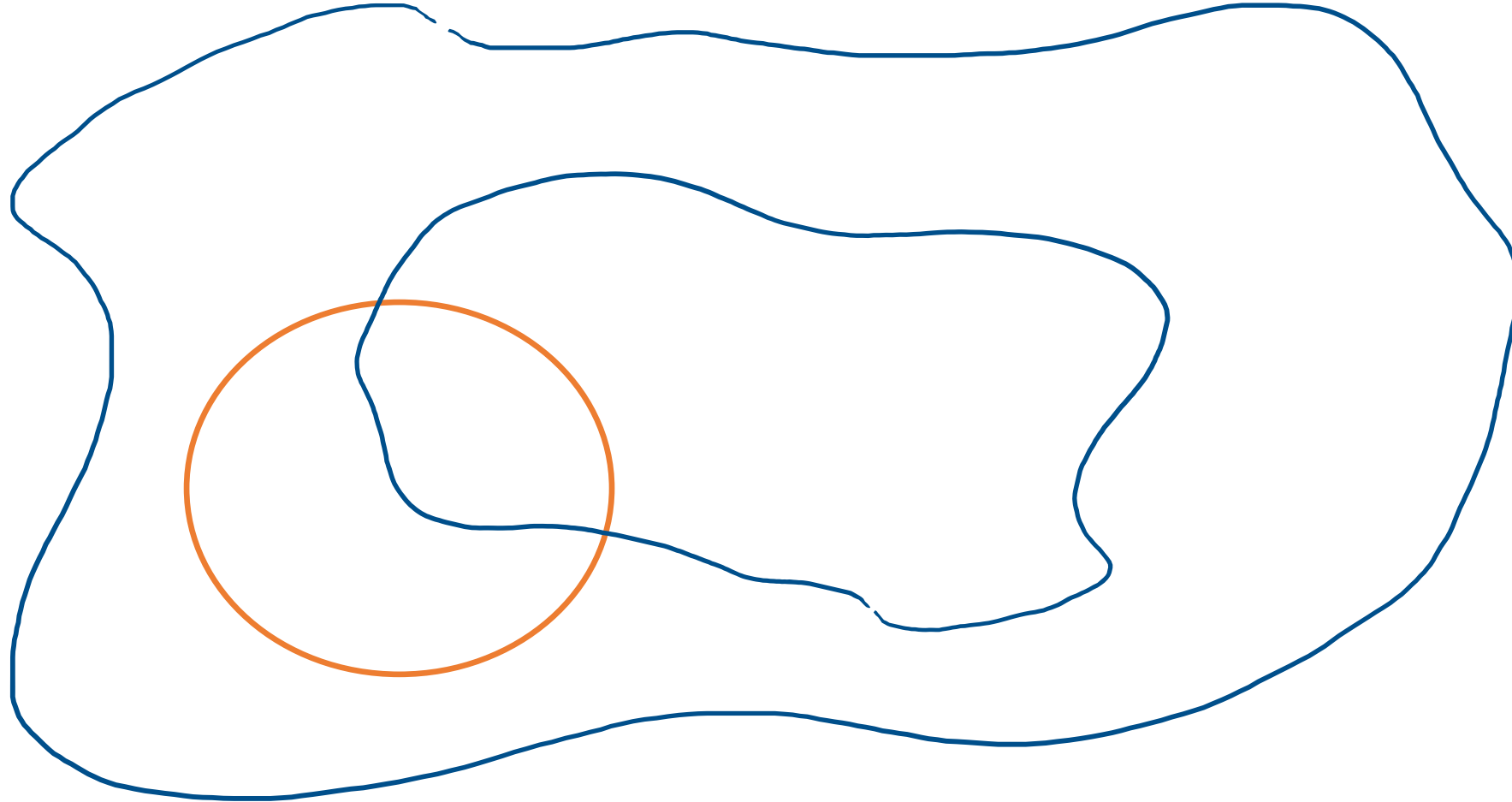
# Posterior



# Markov chain Monte Carlo (MCMC)

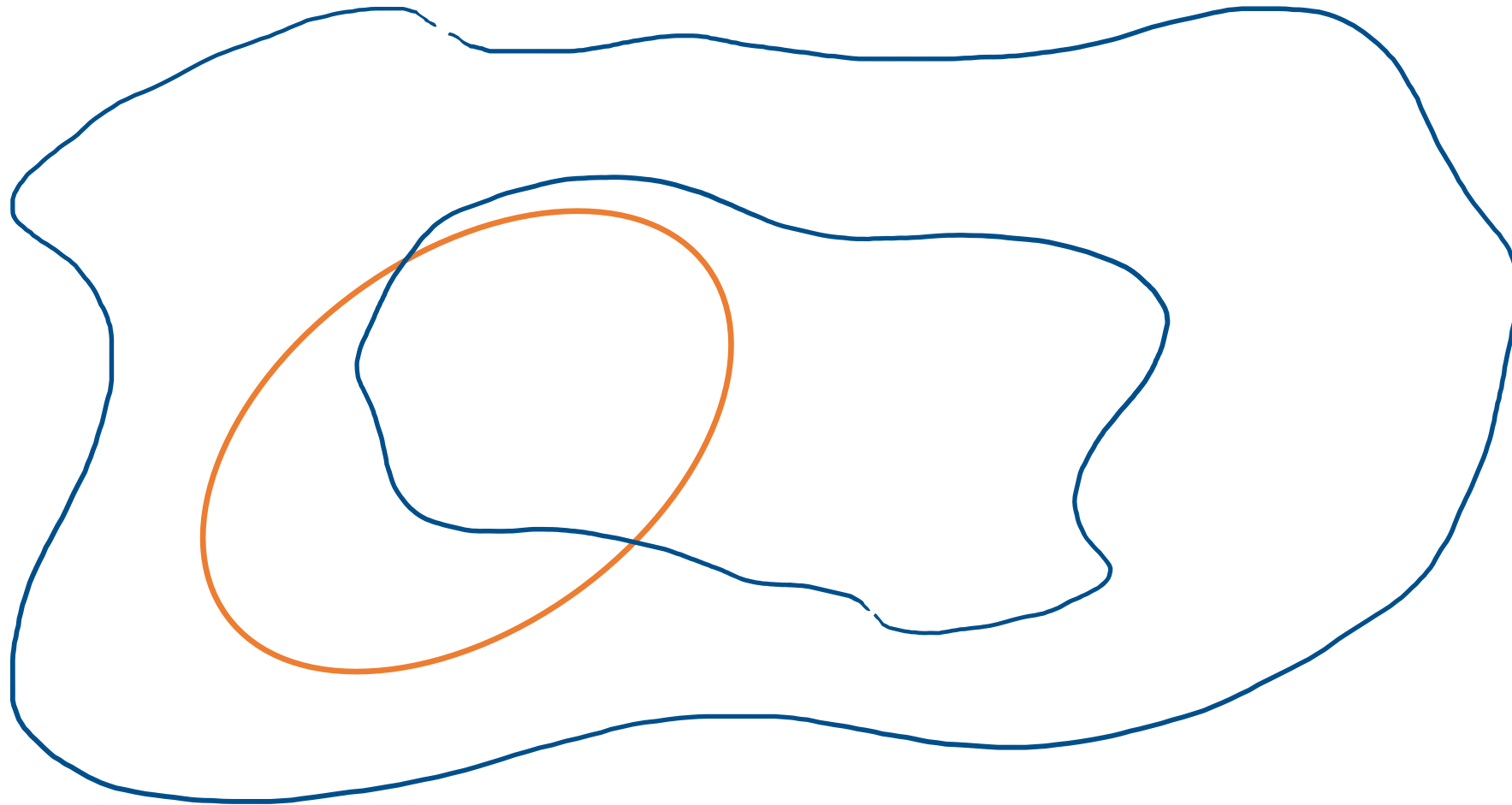


# Variational Inference (VI)



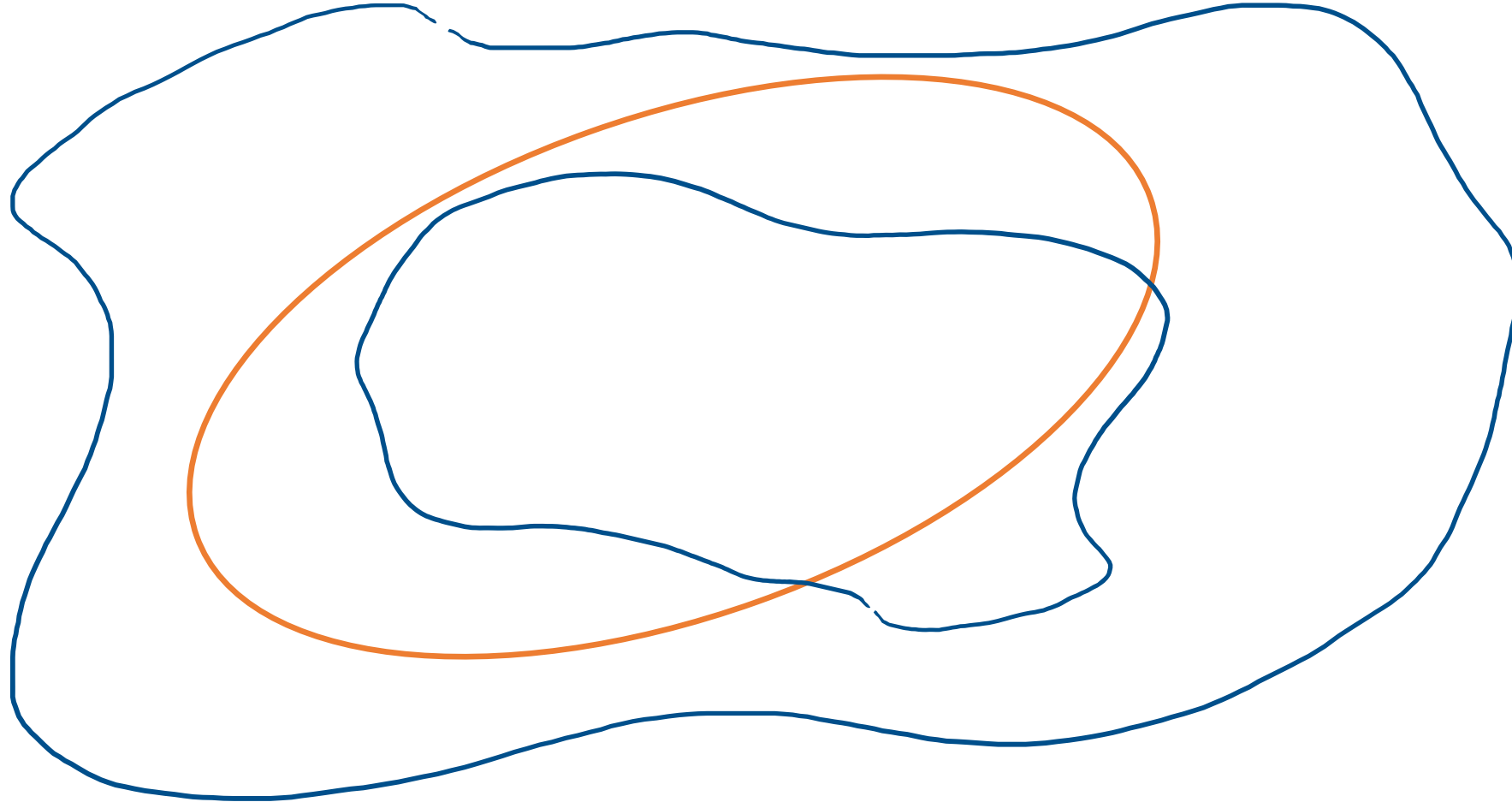
$$\min_{q \in \text{Family}} \text{KL}(q || p)$$

# Variational Inference (VI)



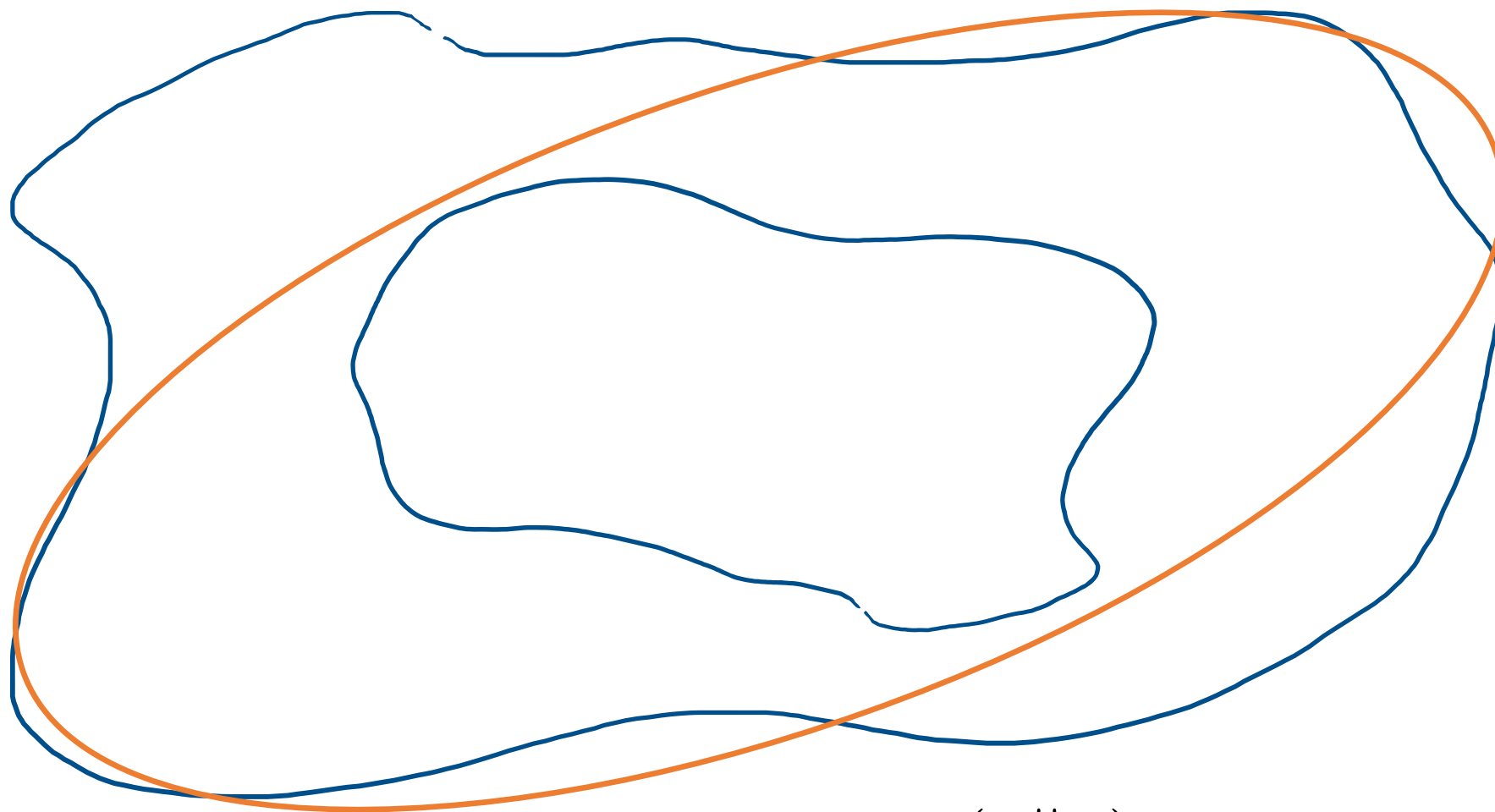
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# Variational Inference (VI)



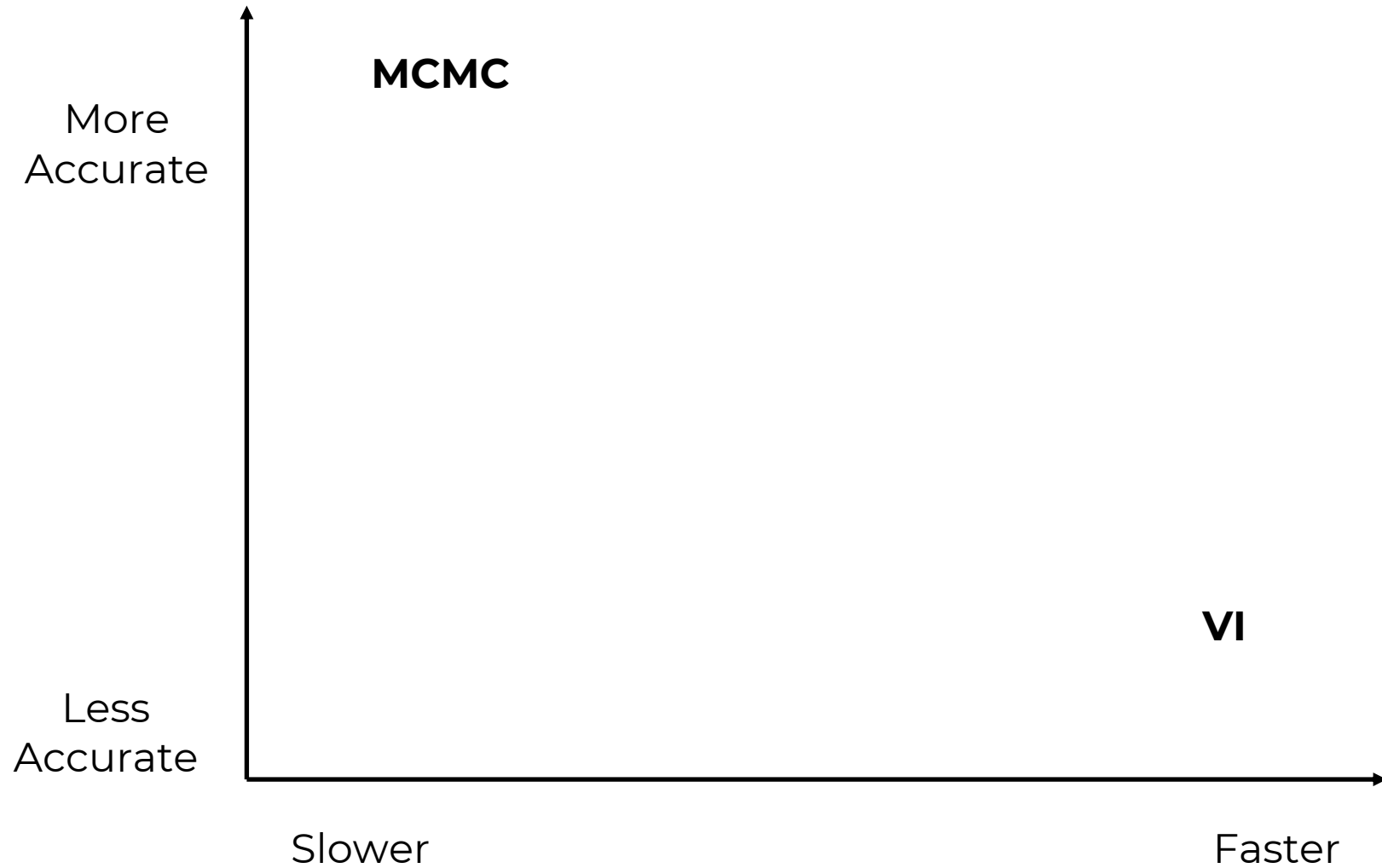
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# Variational Inference (VI)

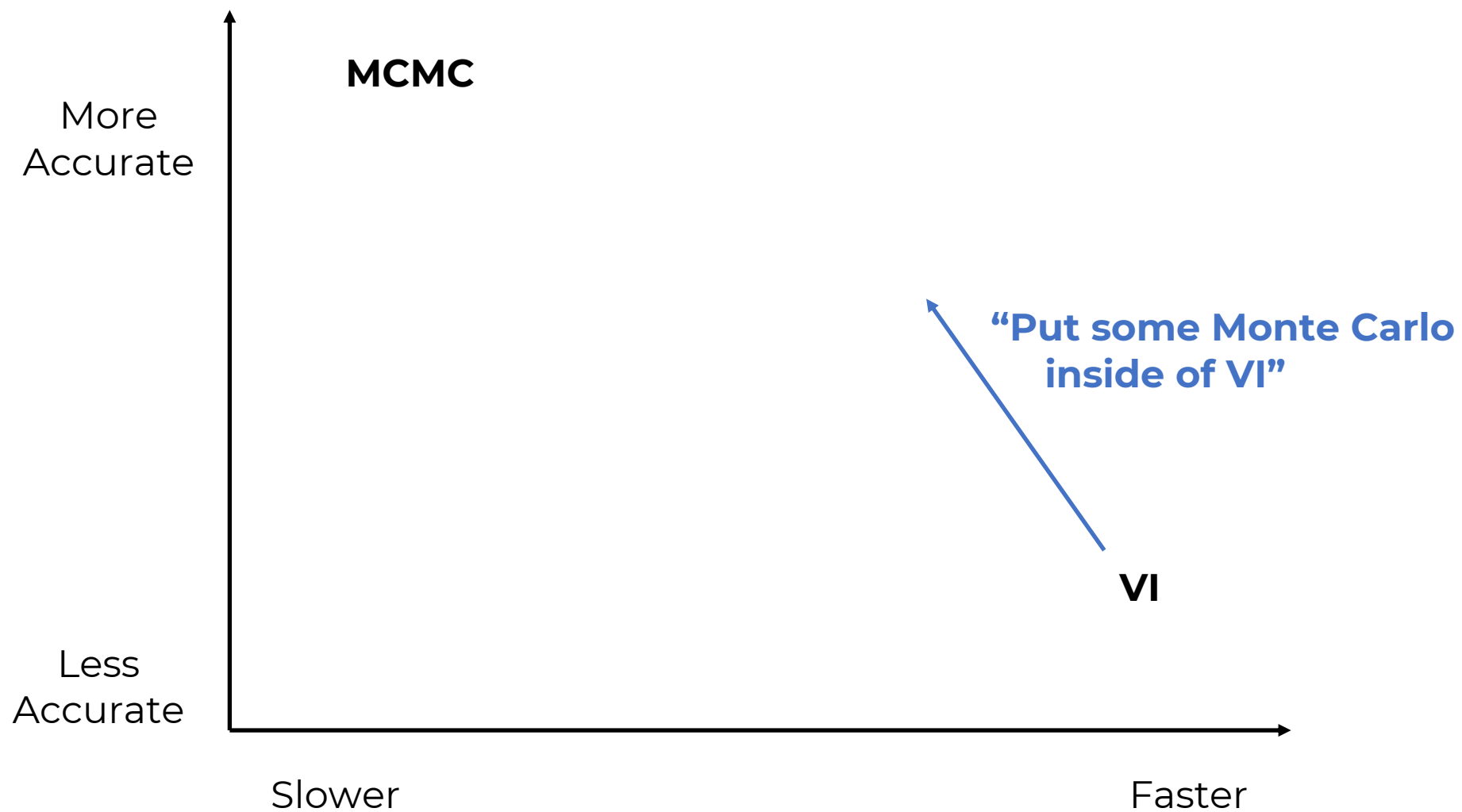


$$\min_{q \in \text{Family}} \text{KL}(q || p)$$

# Folk Wisdom



# This Talk



# Outline

Importance Weighted VI

Variational Particle Filters

Variational MCMC

# Importance Weighted VI

(Plus {Antithetic, Stratified, Quasi-Monte Carlo, Latin hypercube} VI)

# The inference problem

Target distribution:  $p(z, x)$

data:  $x$  (seen)

latent variables:  $z$  (unseen)

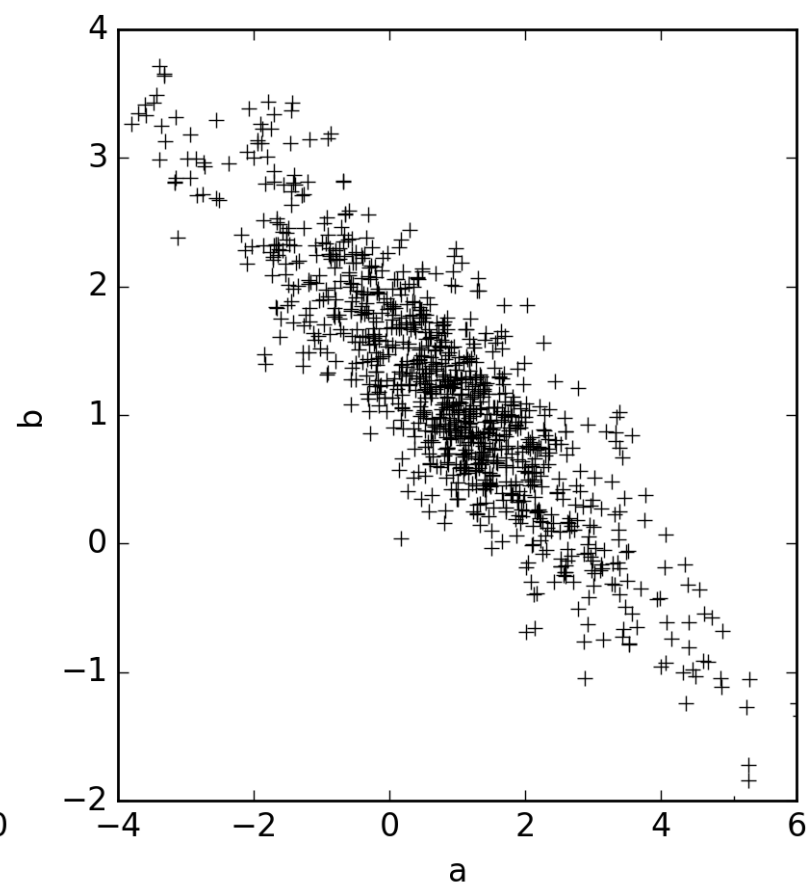
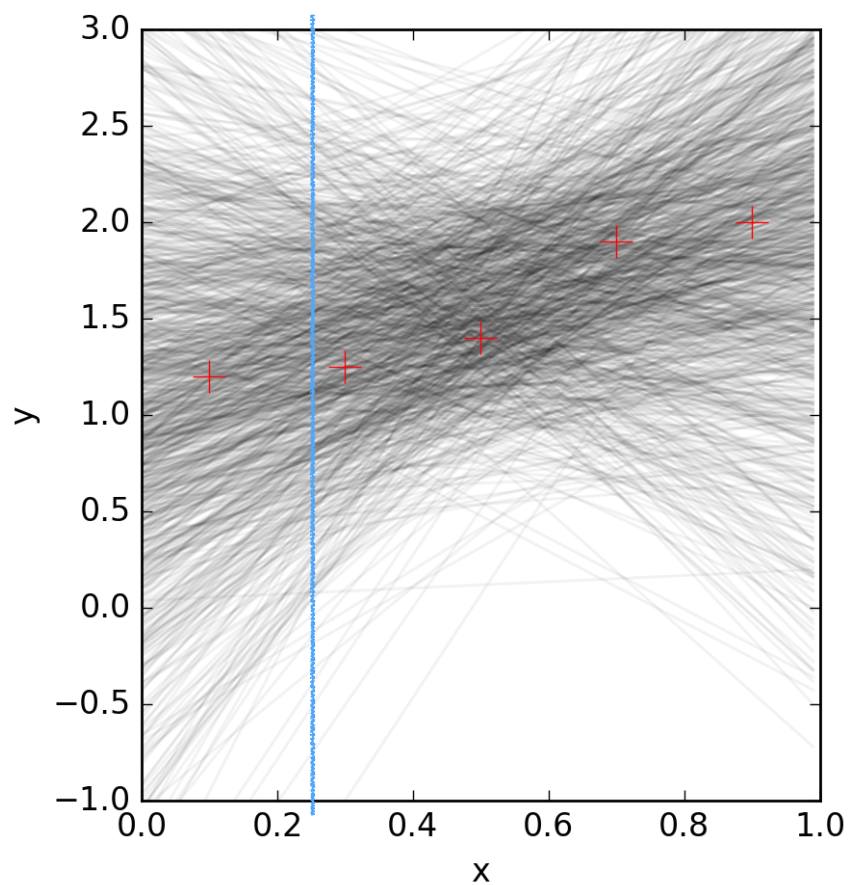
## **Goals:**

Bayesian inference: Approximate  $p(z|x)$

Learning: Maximize  $\log p(x)$

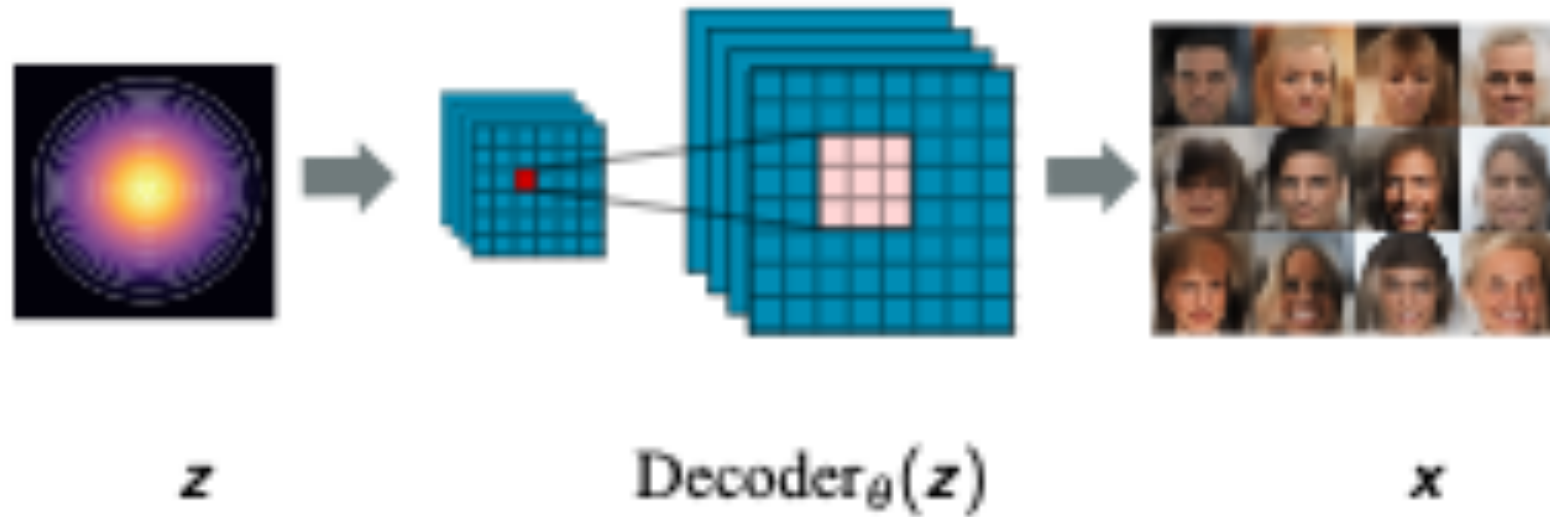
# Bayesian regression

$$p(a, b|x, y) \propto p(a, b) \prod_i p(y_i|x_i, a, b)$$



# Learning: Variational auto-encoder

$$p_{\theta}(z, x) = p(z)p_{\theta}(x|z)$$



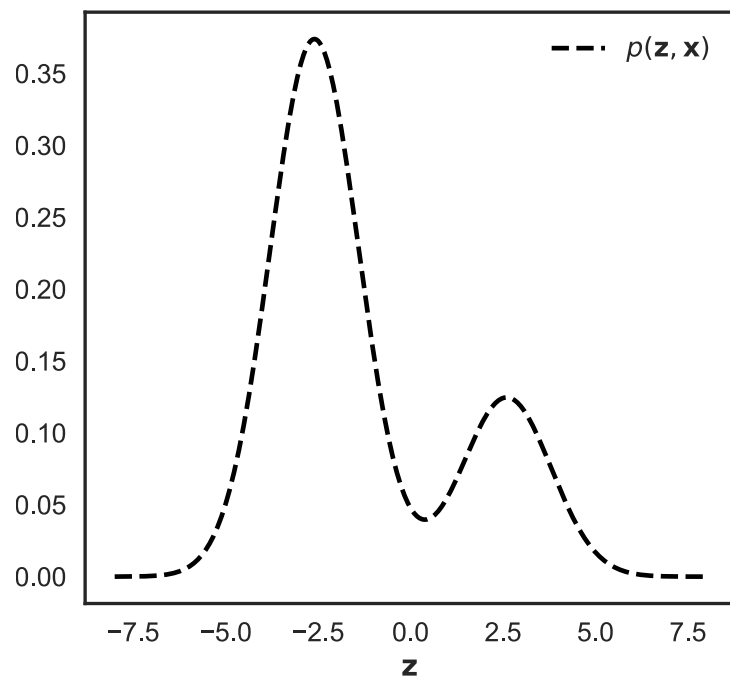
Use VI to bound  $\log p_{\theta}(x)$

# The “ELBO”

is p high?

is q spread?

$$\text{ELBO}(q||p) = \mathbb{E}_{q(z)} [\log p(z, x) - \log q(z)]$$

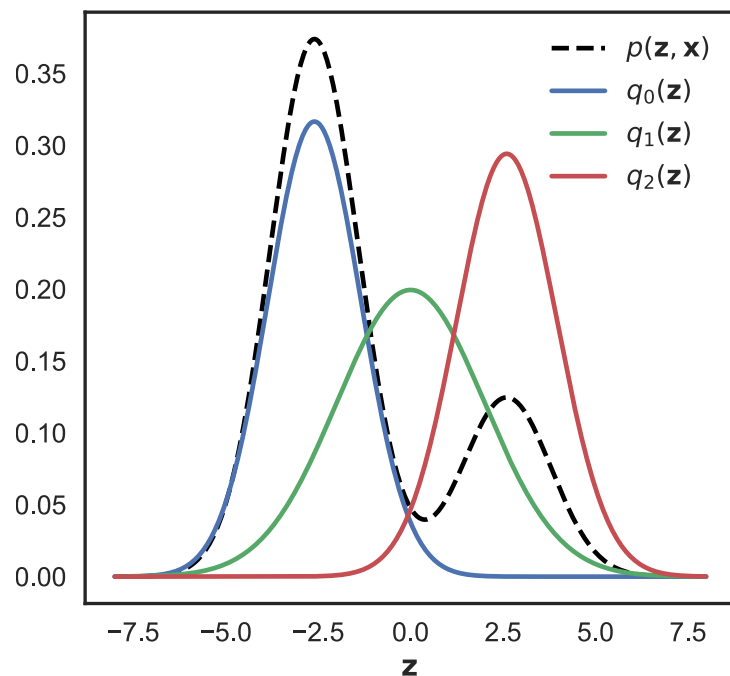


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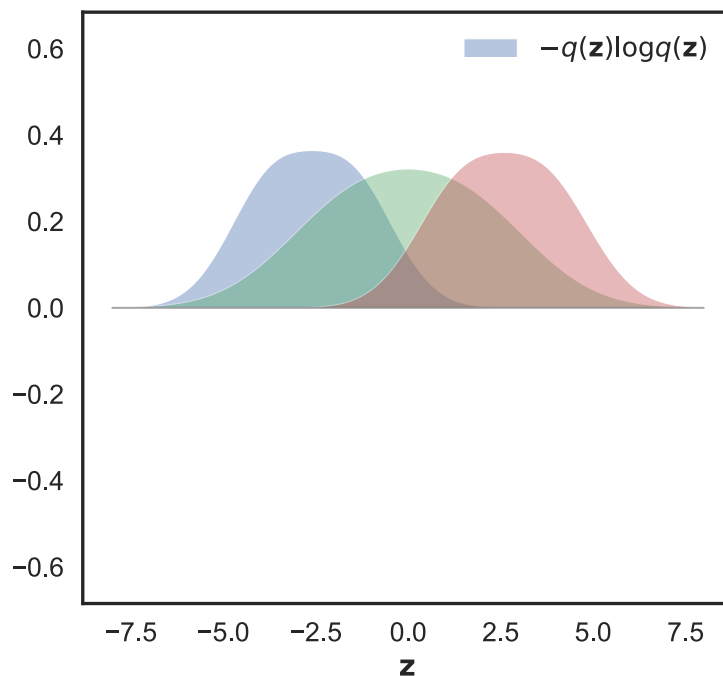
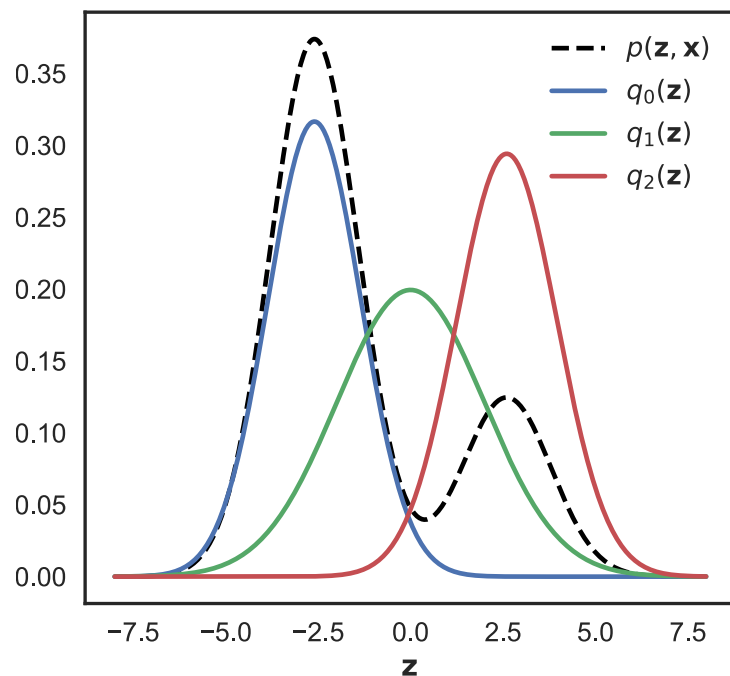


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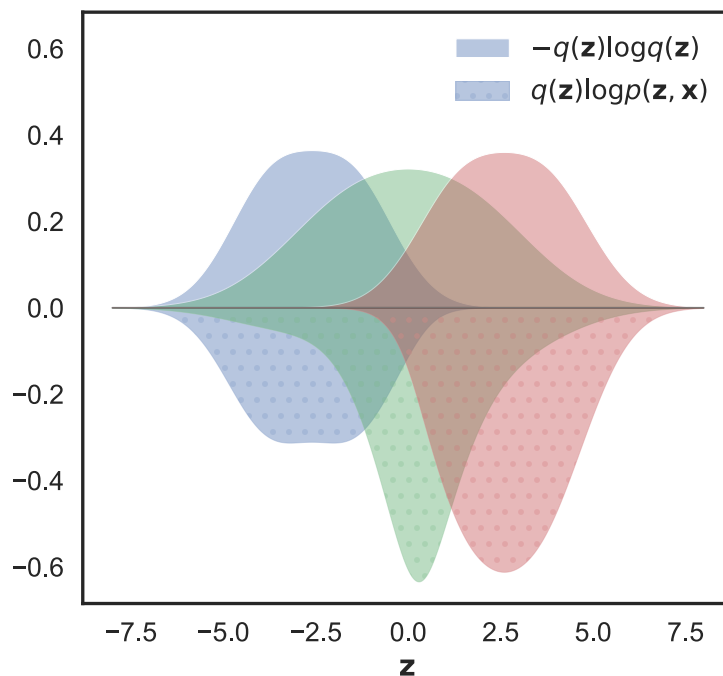
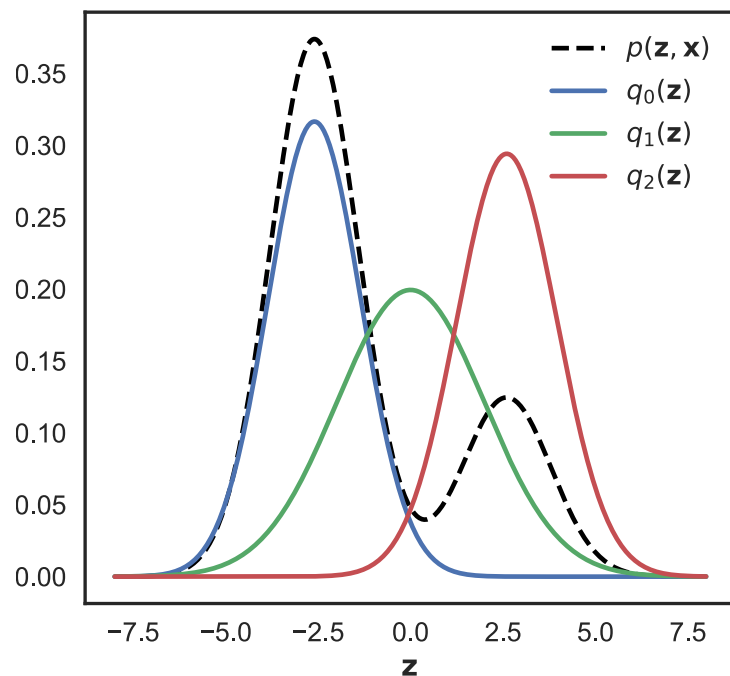


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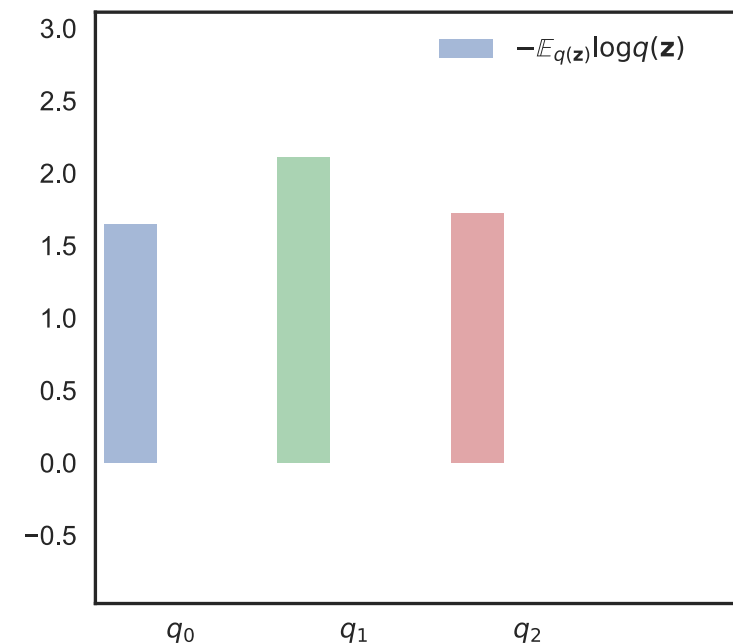
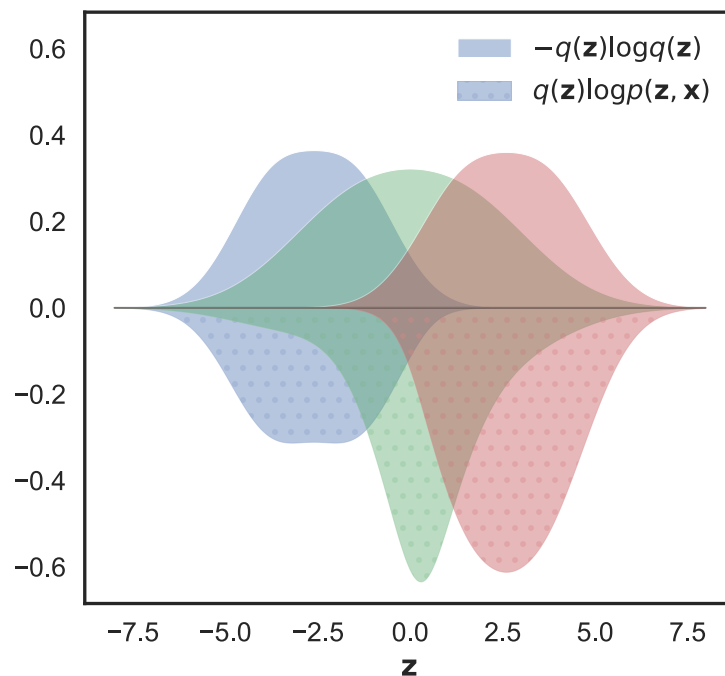
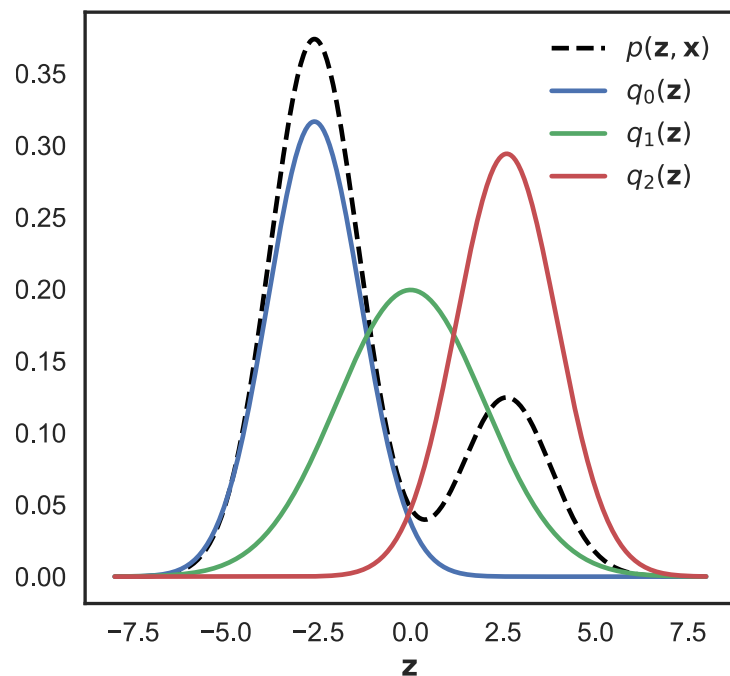


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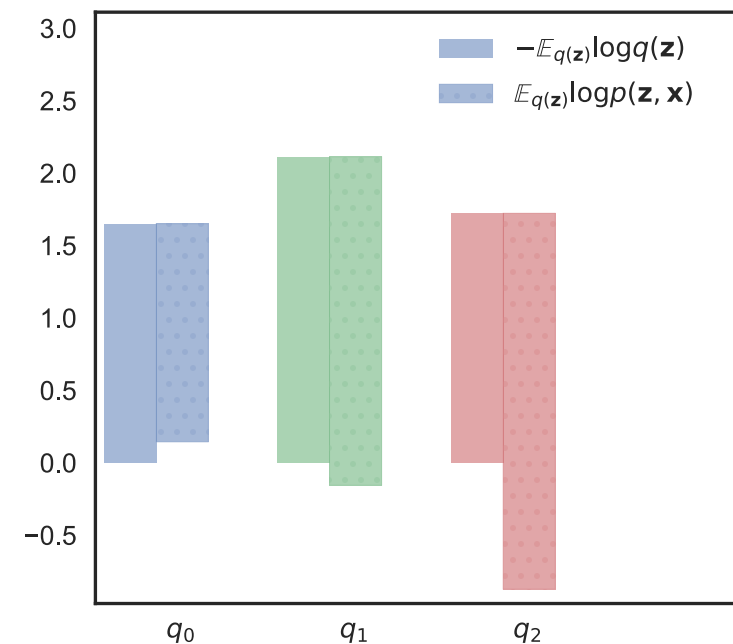
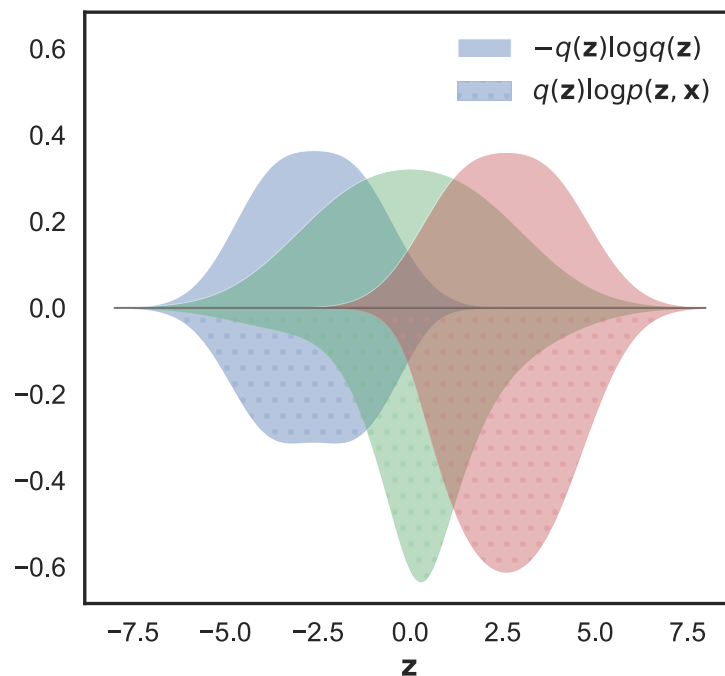
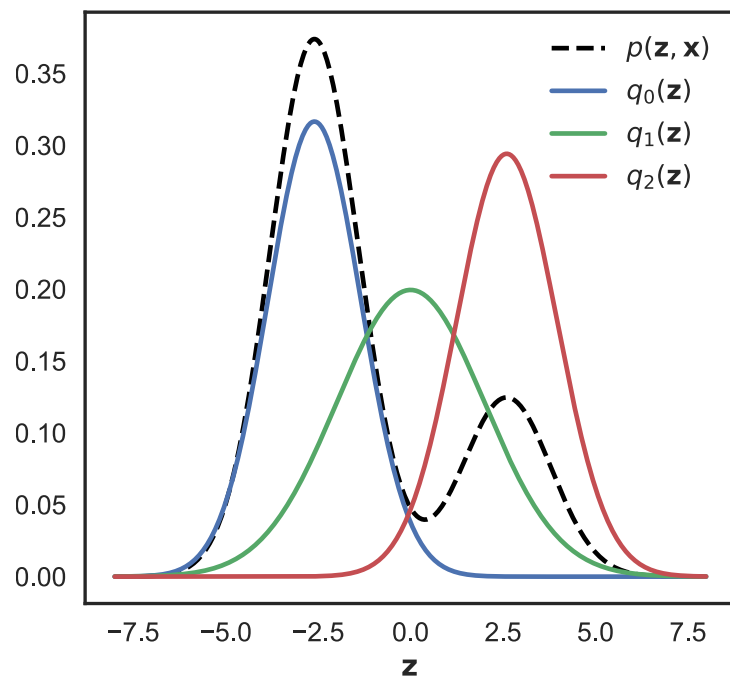


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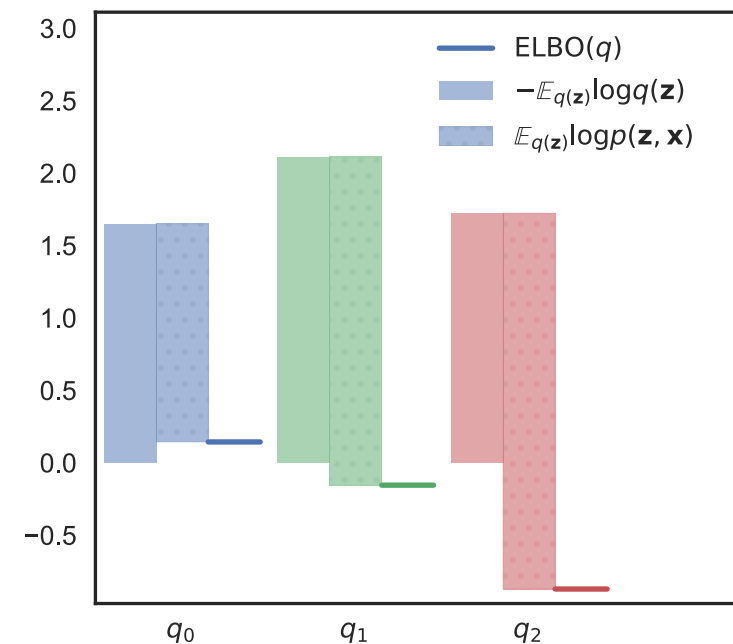
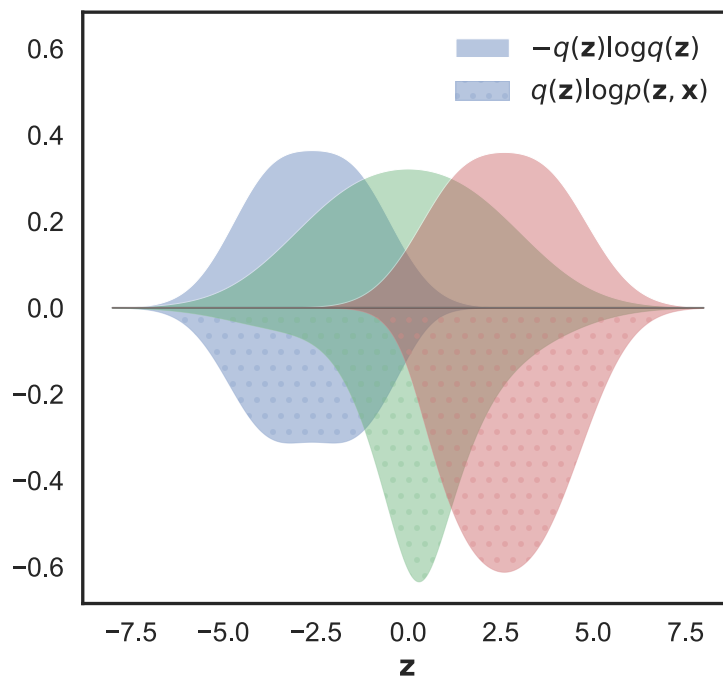
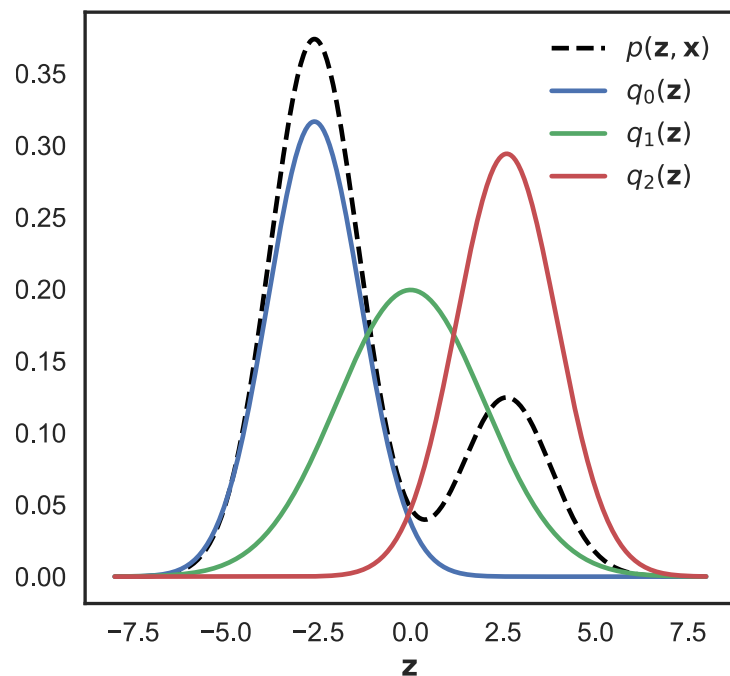


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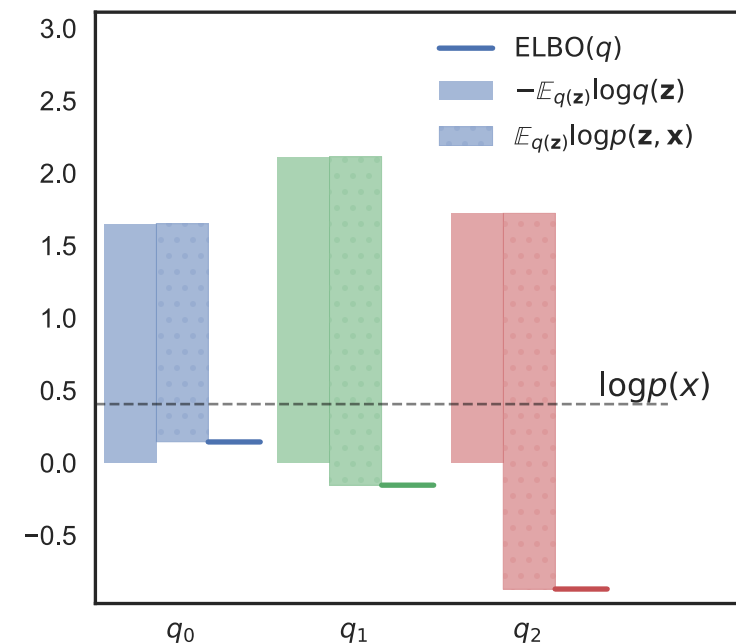
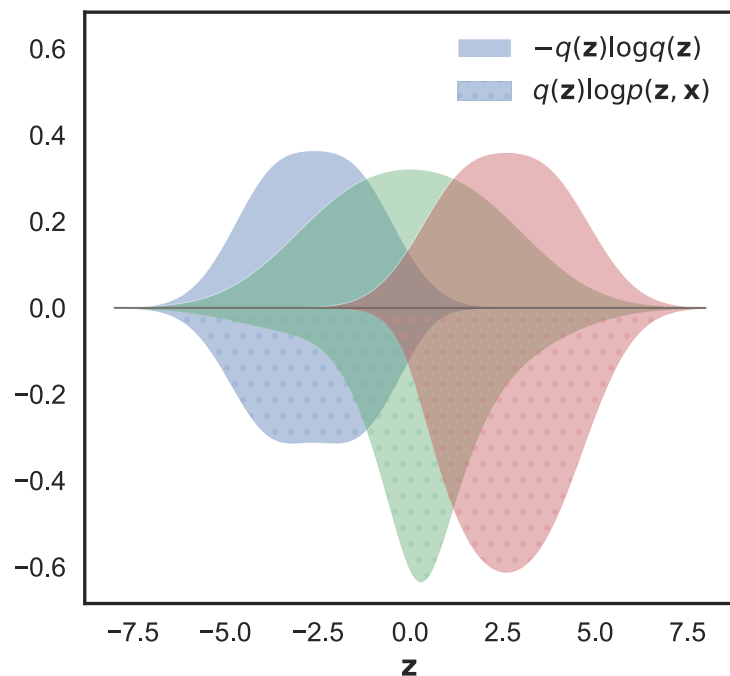
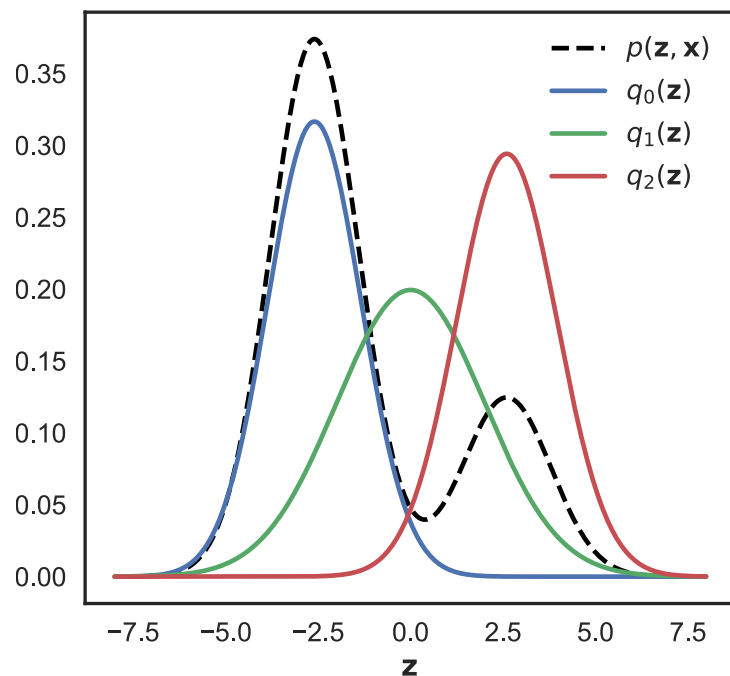


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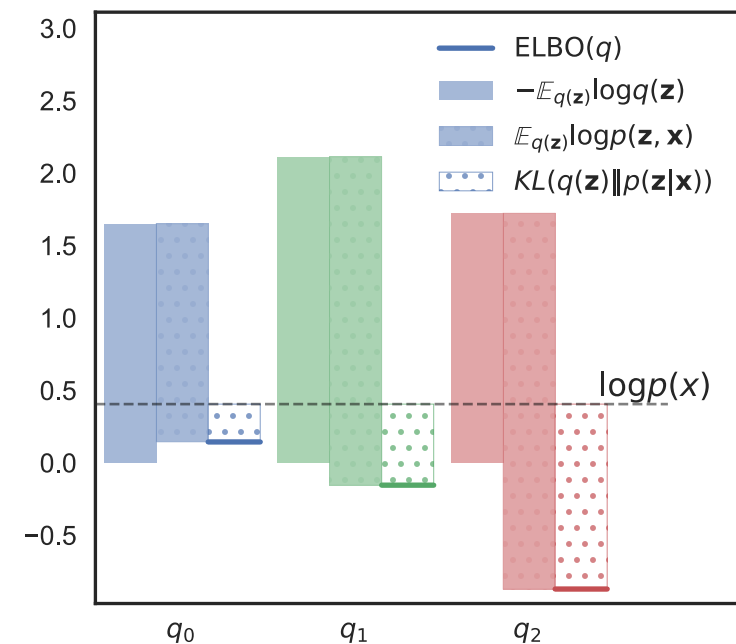
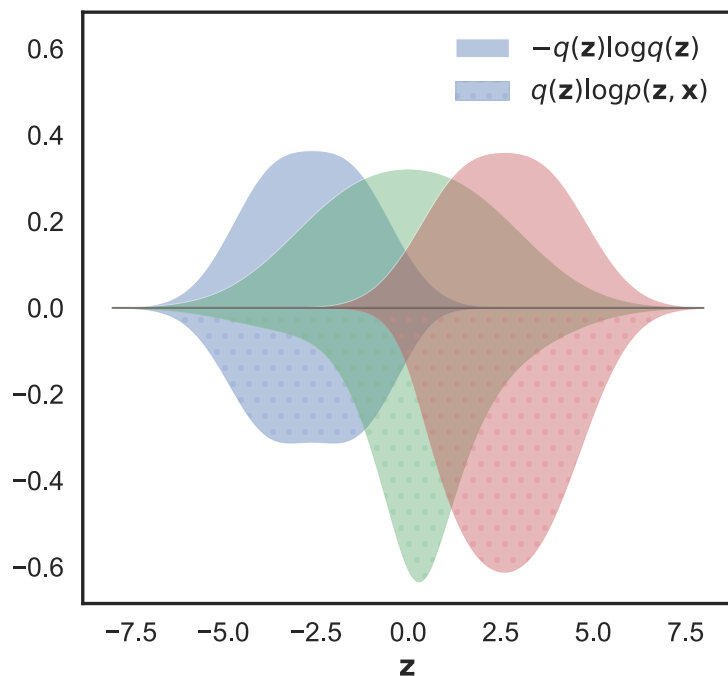
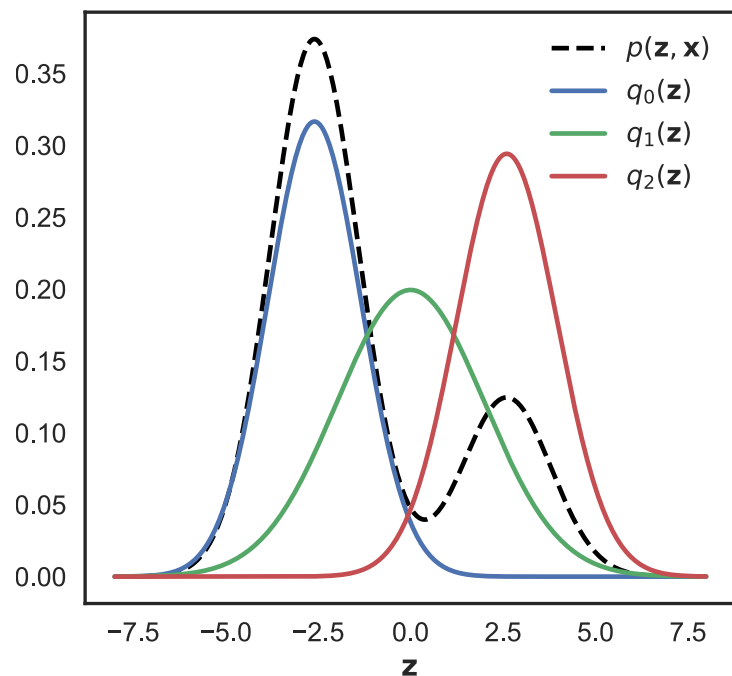


# The “ELBO”

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$$\text{ELBO}(q||p) = \mathbb{E}_{q(z)} [\log p(z, x) - \log q(z)]$$



# ELBO decomposition

$$\log p(x) = \underbrace{\mathbb{E}_{q(z)} [\log p(z, x) - \log q(z)]}_{\text{ELBO}(q||p)} + \text{KL}(q(z)||p(z|x))$$

Lower bound:  $\text{ELBO}(q||p) \leq \log p(x)$

Looseness:  $\text{KL}(q(z)||p(z|x))$

# Thought experiment

If  $\mathbb{E}R = p(x)$  and  $R > 0$ ,

$$\log p(x) = \underbrace{\mathbb{E} \log R}_{\text{bound}} + \underbrace{\mathbb{E} \log \frac{p(x)}{R}}_{\text{looseness}}$$

If  $R = \frac{p(z, x)}{q(z)}$ ,  $z \sim q$ , becomes ELBO decomposition.

But you can use any  $R$ ....

# Variational Autoencoders

$$R = \frac{p_{\theta}(z, x)}{q(z)}, z \sim q$$

Input  $p_{\theta}(x, z)$  and  $x$ .

Find some  $q(z)$  and  $\theta$  to make  $\mathbb{E} \log R$  large.

Output  $\theta$ .

# Importance Weighted Autoencoders

$$R_M = \frac{1}{M} \sum_{m=1}^M \frac{p_{\theta}(z_m, x)}{q(z_m)}, z_m \sim q$$

Input  $p_{\theta}(x, z)$  and  $x$ .

Find some  $q(z)$  and  $\theta$  to make  $\mathbb{E} \log R_M$  large.

Output  $\theta$ .

# Wait a second...

**Good old VI:**  $\log p(x) = \mathbb{E} \log R + \text{KL}(q(z) || p(z|x))$

Learning:  $\mathbb{E} \log R \leq \log p(x)$

Inference:  $p(z|x) \approx q(z)$

**Now:**  $\log p(x) = \mathbb{E} \log R_M + \mathbb{E} \log \frac{p(x)}{R_M}$

Learning:  $\mathbb{E} \log R_M \leq \log p(x)$

Inference:  $p(z|x) \approx ???$

# Self-Normalized Importance Sampling

$q_M(z_1) :$

$$\hat{z}_1, \dots, \hat{z}_M \sim q(z)$$

Sample  $m \in \{1, \dots, M\}$  with  $\mathbb{P}(m) \propto p(\hat{z}_m, x) / q(\hat{z}_m)$

Return  $z_1 = \hat{z}_m$

$$\text{KL}(q_M(z) || p(z|x)) \longleftarrow \text{intractable}$$

# Importance weighted VI

$q_M(\textcolor{red}{z_1}) :$

$$\hat{z}_1, \dots, \hat{z}_M \sim q(z)$$

Sample  $m \in \{1, \dots, M\}$  with  $\mathbb{P}(m) \propto p(\hat{z}_m, x) / q(\hat{z}_m)$

Return  $\textcolor{red}{z_1} = \hat{z}_m$

# Importance weighted VI

$q_M(z_{1:M}) :$

$$\hat{z}_1, \dots, \hat{z}_M \sim q(z)$$

Sample  $m \in \{1, \dots, M\}$  with  $\mathbb{P}(m) \propto p(\hat{z}_m, x) / q(\hat{z}_m)$

Return  $z_1 = \hat{z}_m$  and  $z_{2:M} = \hat{z}_{-m}$

# Importance weighted VI

$q_M(z_{1:M}) :$

$\hat{z}_1, \dots, \hat{z}_M \sim q(z)$

Sample  $m \in \{1, \dots, M\}$  with  $\mathbb{P}(m) \propto p(\hat{z}_m, x) / q(\hat{z}_m)$

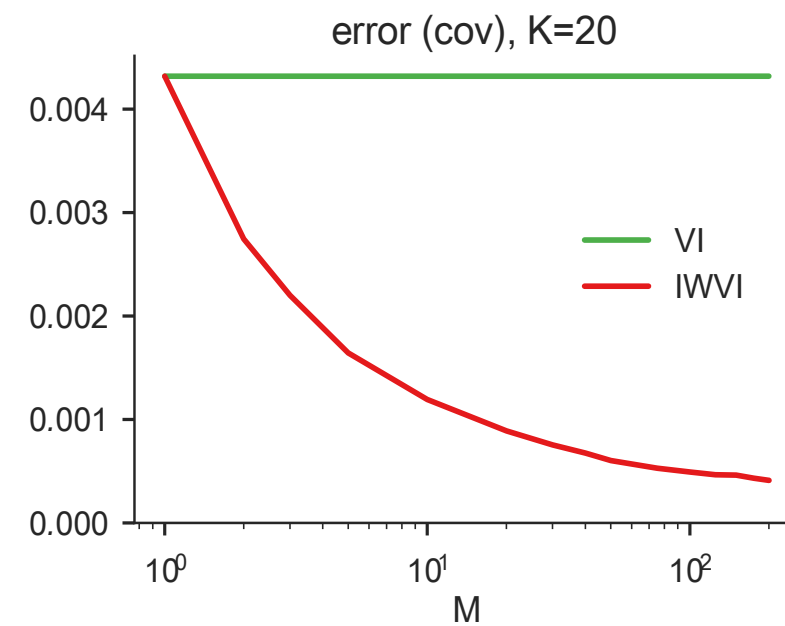
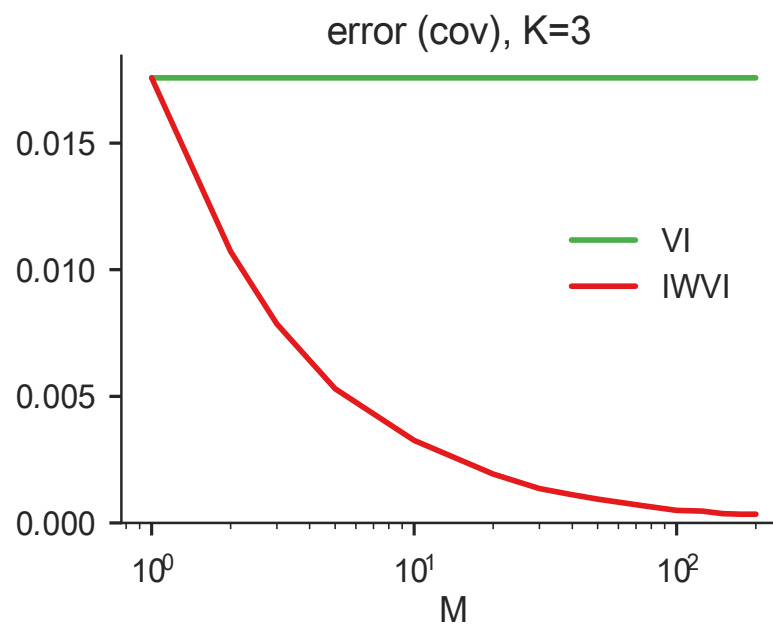
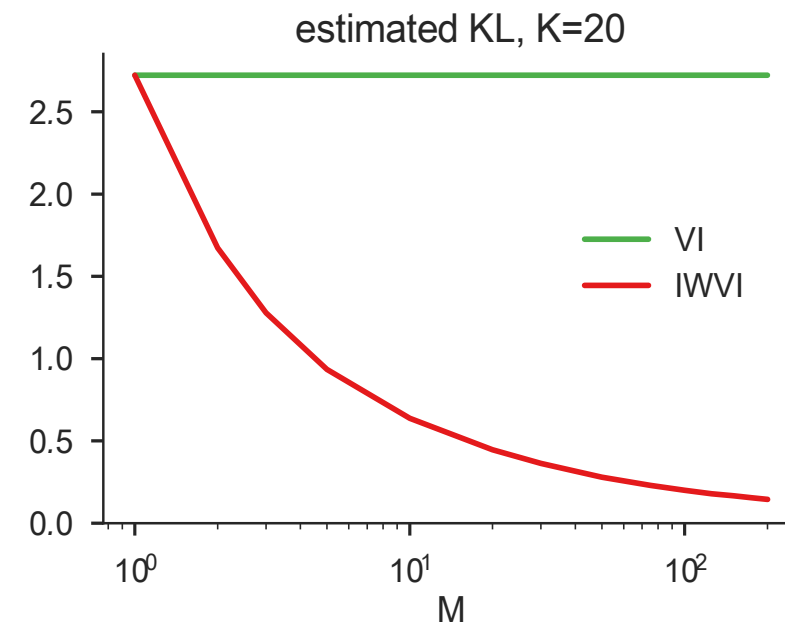
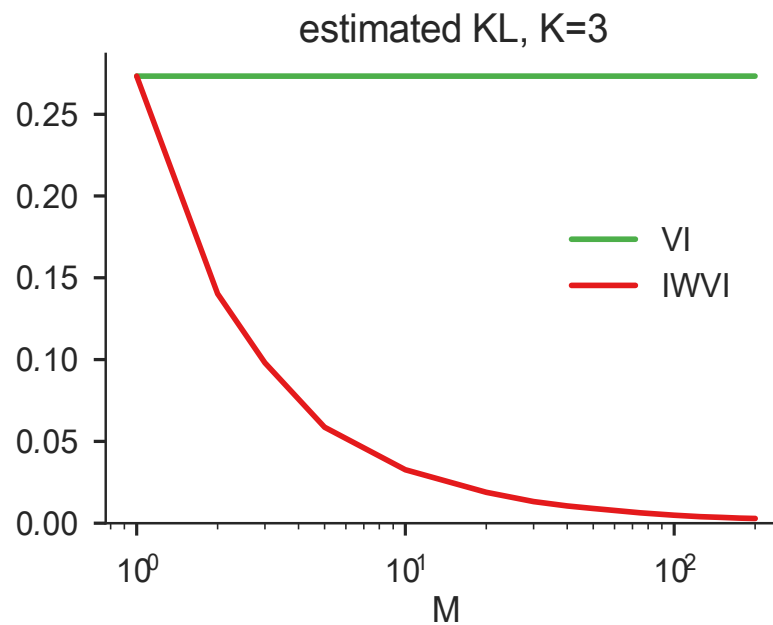
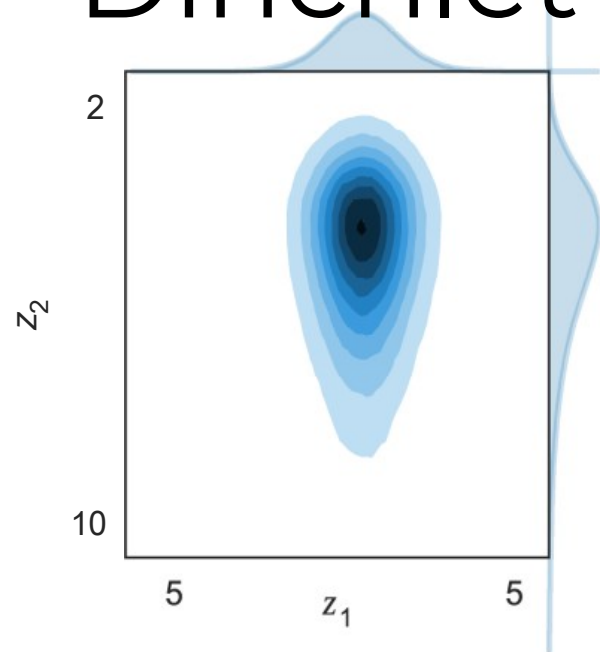
Return  $z_1 = \hat{z}_m$  and  $z_{2:M} = \hat{z}_{-m}$

$p_M(z_{1:M}, x) := p(z_1, x) q(z_2) \cdots q(z_M)$

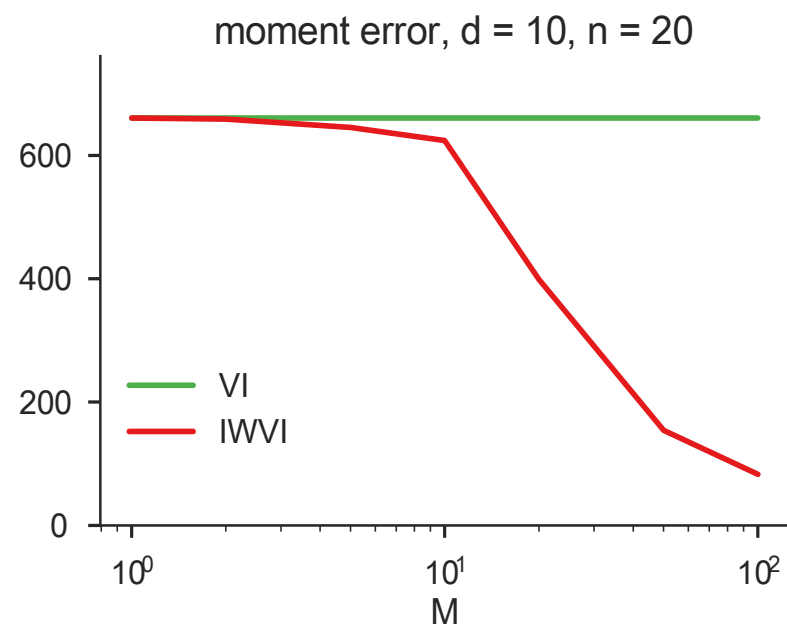
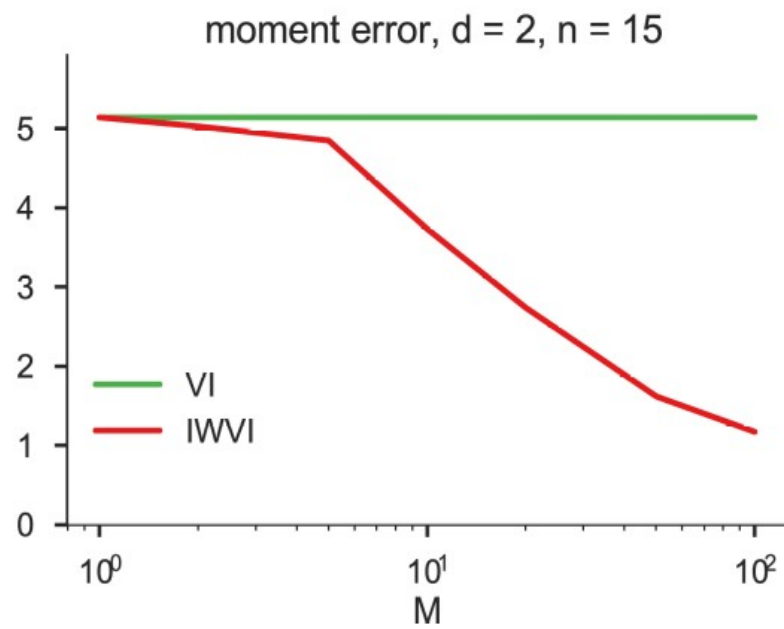
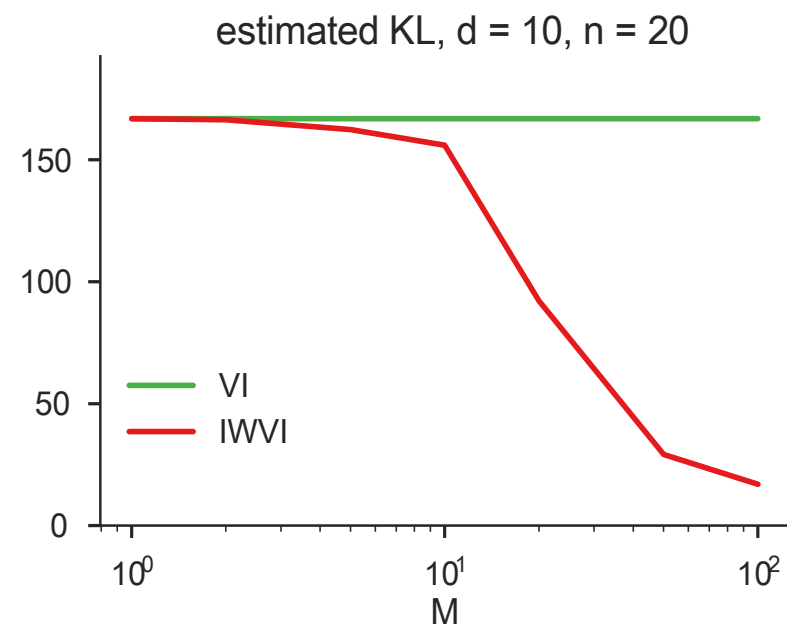
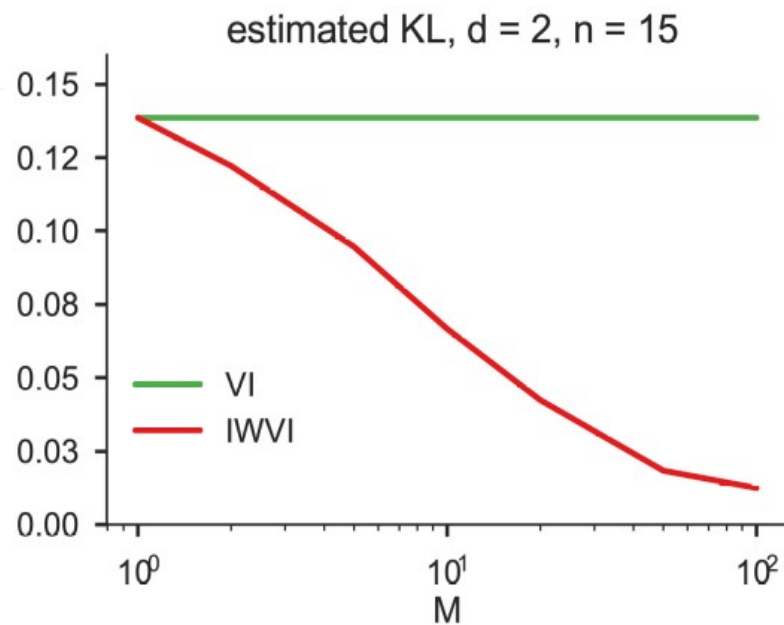
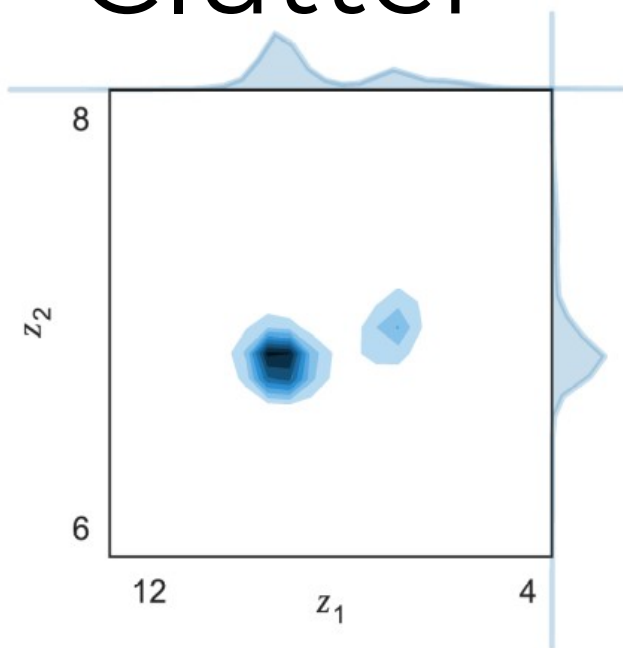
$$\log p(x) = \mathbb{E} \log R_M + \text{KL}(q_M(z_{1:M}) || p_M(z_{1:M} | x))$$

(Bachman & Precup 2015, Cremer et al., 2017, Naesseth et al. 2018, Le et al. 2018, D. & Sheldon, 2018)

# Dirichlet



# Clutter



# Huh?

$$\mathbb{E} R = p(x) \implies \log p(x) = \mathbb{E} \log R + \mathbb{E} \log \frac{p(x)}{R}$$

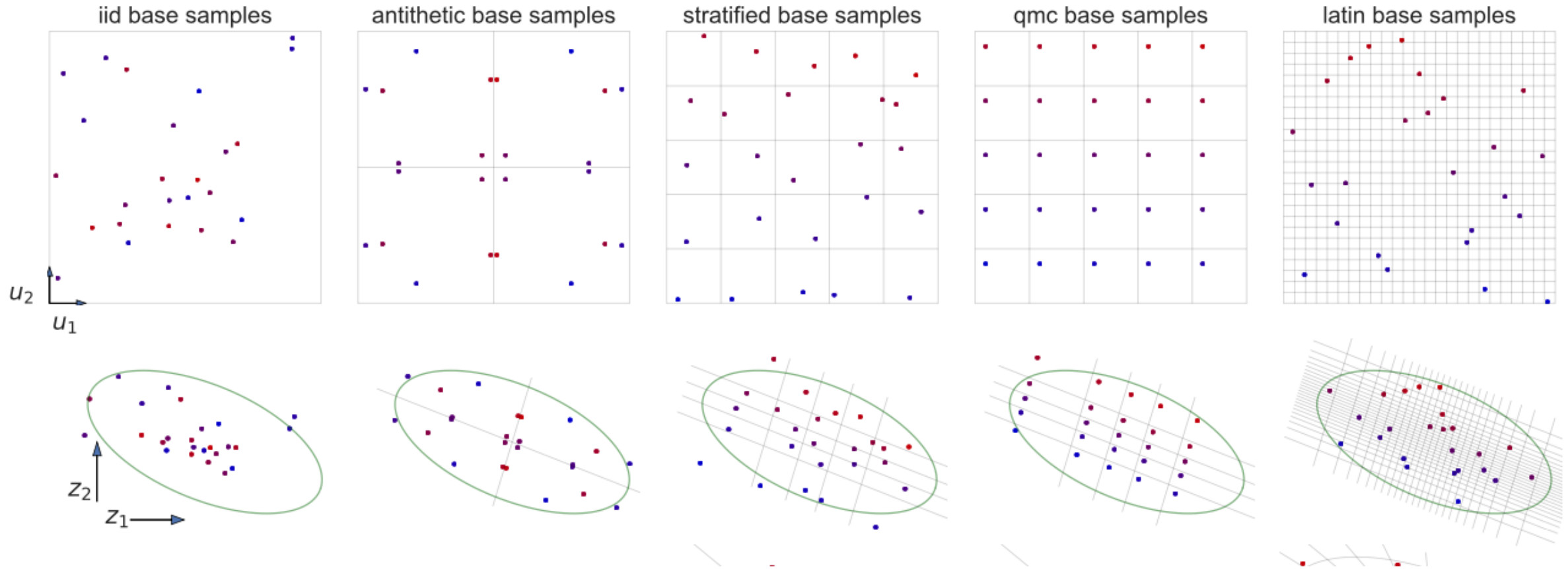
$$R = \frac{p(z, x)}{q(z)} \implies \log p(x) = \mathbb{E} \log R + \mathbb{E} \log \frac{p(z|x)}{q(z)}$$

$$R_M = \frac{1}{M} \sum_{m=1}^M \frac{p(z_m, x)}{q(z_m)} \implies \log p(x) = \mathbb{E} \log R_M + \mathbb{E} \log \frac{p_M(z_{1:M}|x)}{q_M(z_{1:M})}$$

augmented  
distribution?

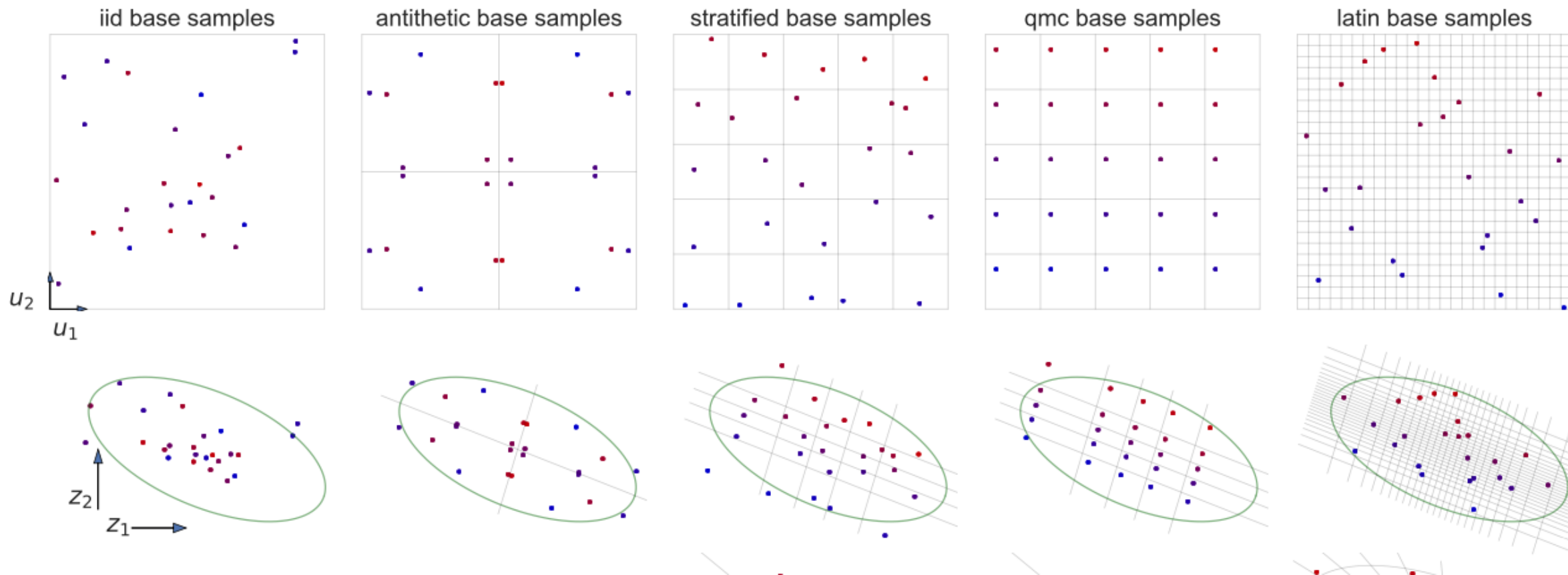
*self-normalized*  
importance sampling!?

# Other estimators?



$$R_M = \frac{1}{M} \sum_{m=1}^M \frac{p(z_m, x)}{q(z_m)}$$

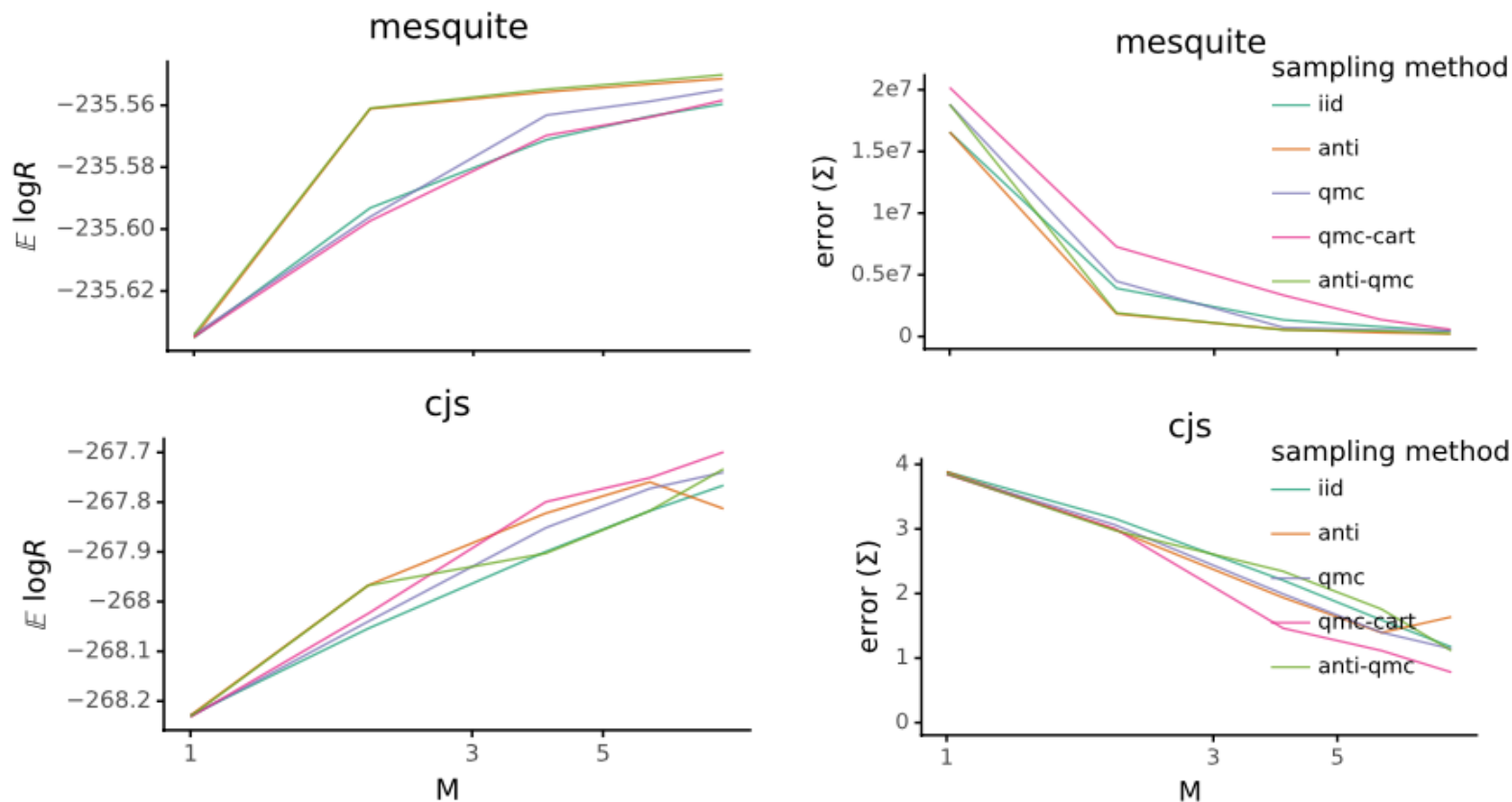
# Divide and couple



Can “Automatically” find posterior approximation via  
“estimator-coupling pairs”

(D. and Sheldon 2019)

# Divide and couple

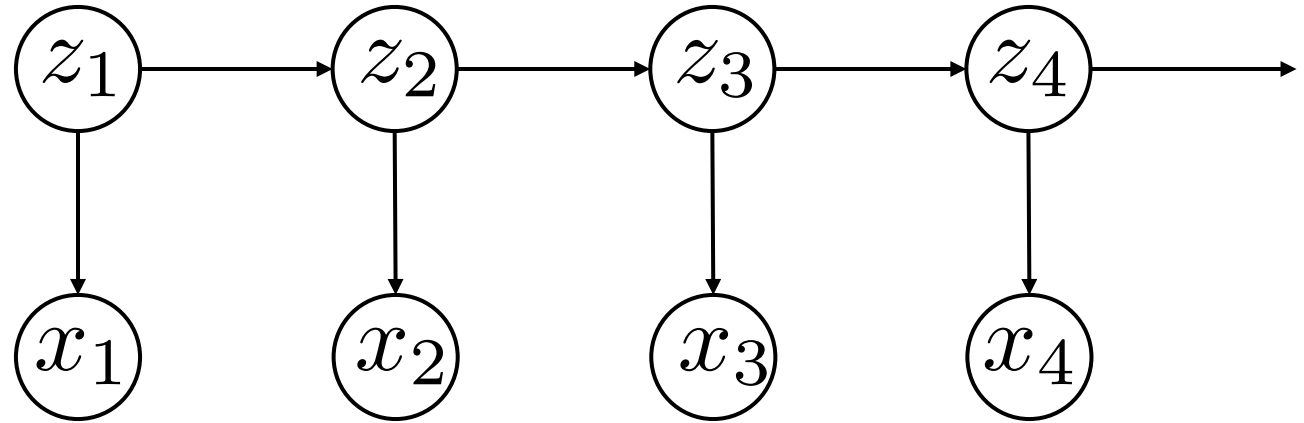


Can “Automatically” find posterior approximation via  
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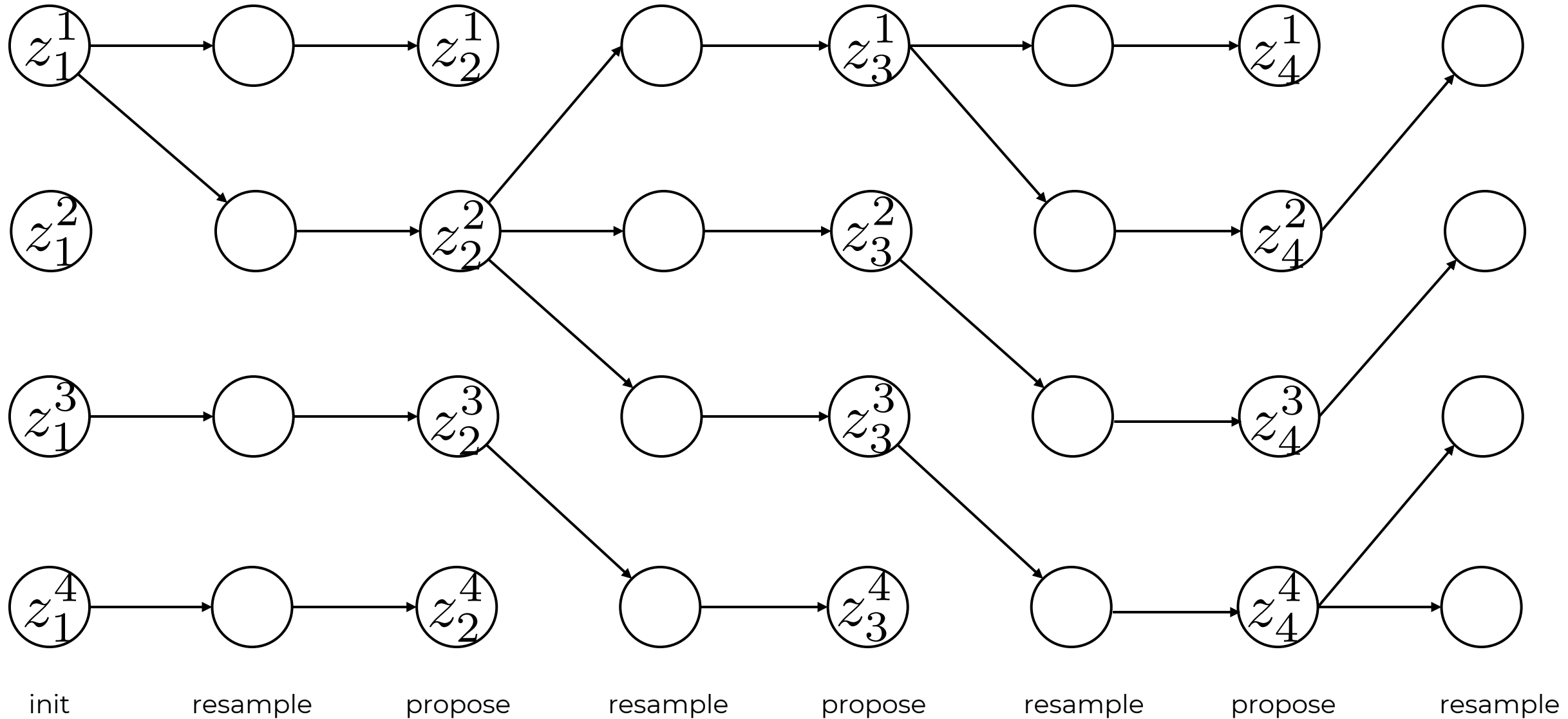
# Variational Particle Filtering

# State-Space Models



$$p(z_{1:T}, x_{1:T}) = p(z_1, x_1) \prod_{t=2}^T p(z_t, x_t | z_{t-1}, x_{t-1})$$

# Particle Filter / Sequential Monte Carlo



# Particle Filter

$$z_1^m \sim q(z_1), \quad w_1^m = p(z_1^m, x_1) / q(z_1^m)$$

For  $t = 2, \dots, T$  :

For  $m = 1, \dots, M$  :

Choose parent  $n \in \{1, \dots, M\}$  with  $\mathbb{P}(n) \propto w_{t-1}^n$

Sample  $z_t^m \sim q(z_t | z_{t-1}^n)$

Set  $w_t^m = p(z_t^m, x_t^m | z_{t-1}^n) / q(z_t^m | z_{t-1}^n)$

# Variational Particle Filter

$$z_1^m \sim q(z_1), \quad w_1^m = p(z_1^m, x_1) / q(z_1^m)$$

For  $t = 2, \dots, T$  :

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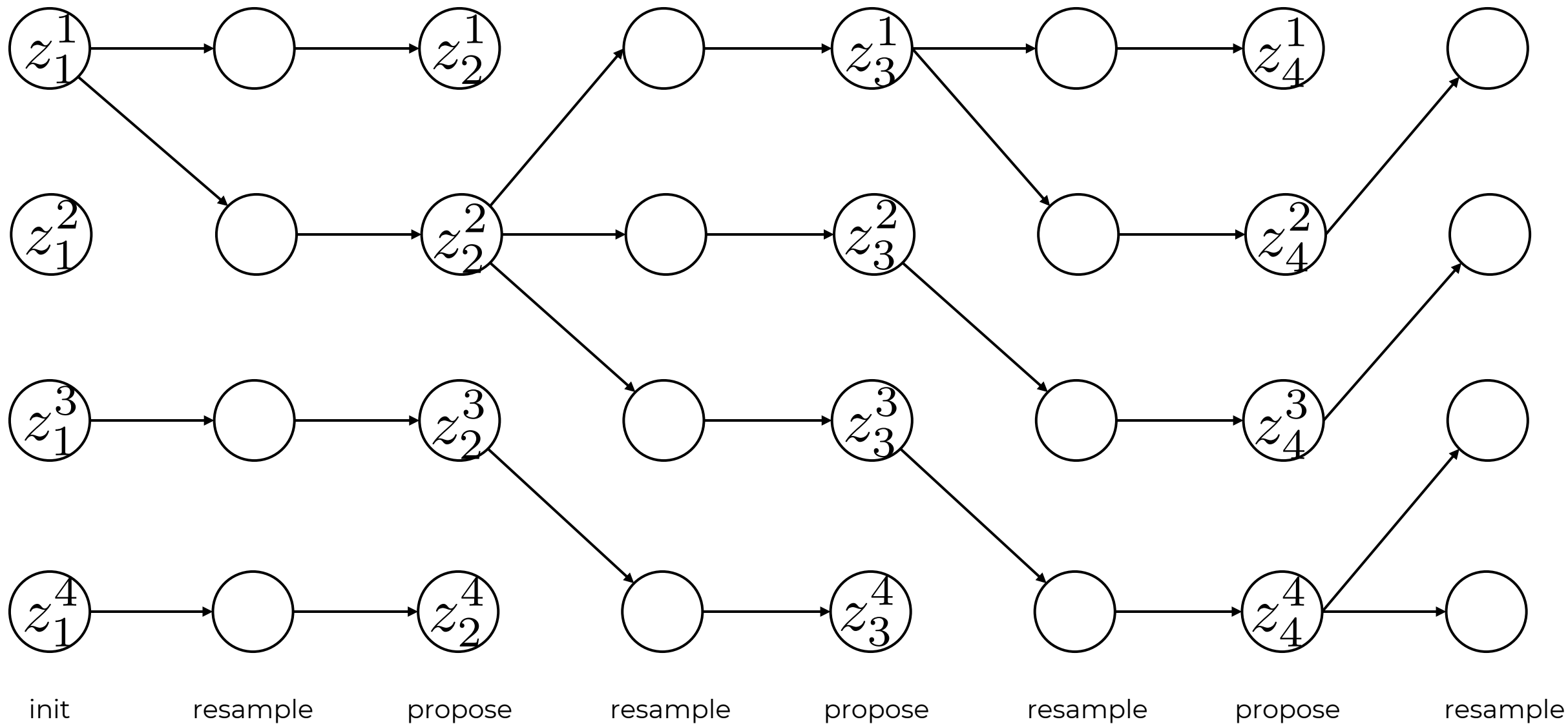
Sample  $z_t^m \sim q(z_t | z_{t-1}^n)$

Set  $w_t^m = p(z_t^m, x_t^m | z_{t-1}^n) / q(z_t^m | z_{t-1}^n)$

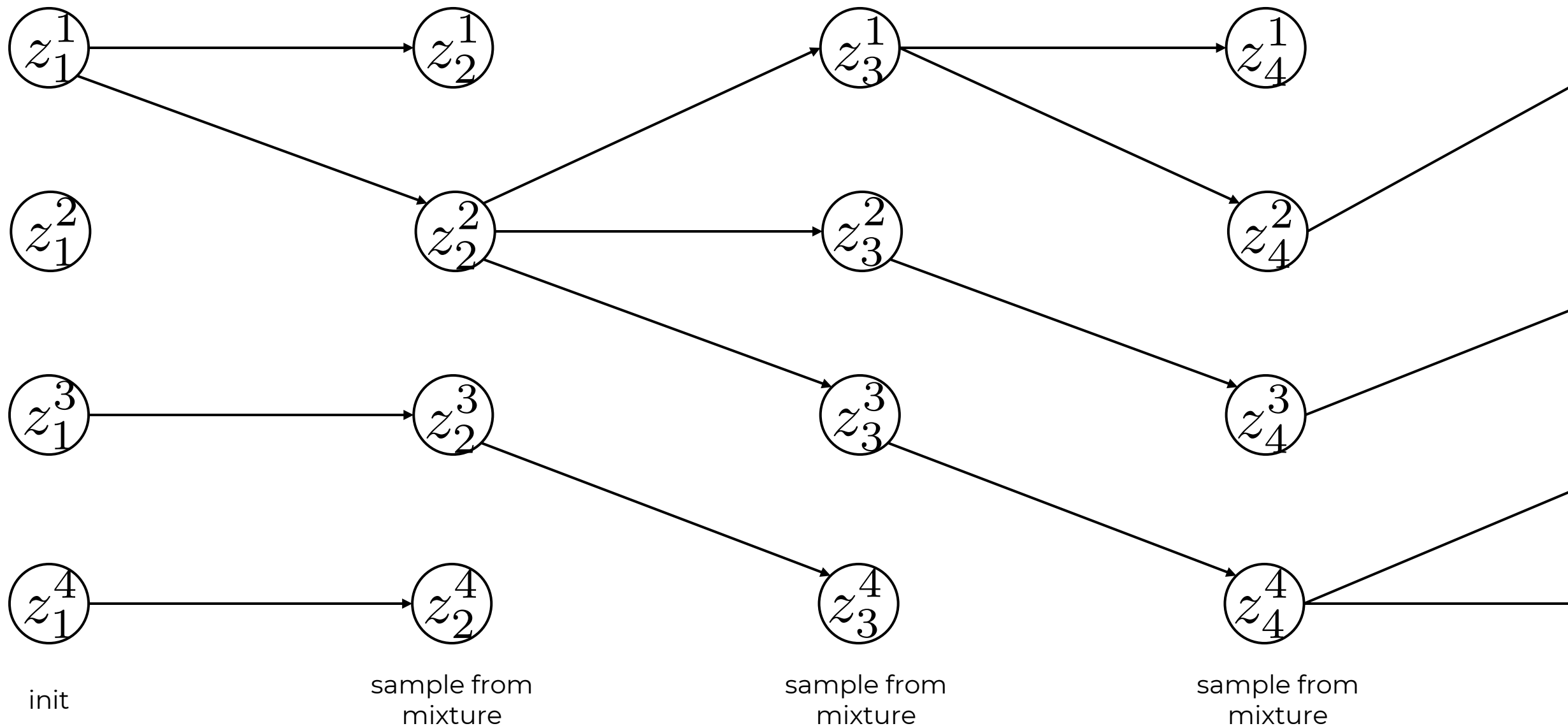
$$R_M := \prod_{t=1}^T \frac{1}{M} \sum_{m=1}^M w_t^m \quad \mathbb{E} R_M = p(x)$$

Maximize  $\mathbb{E} \log R_M$  (Naesseth et al., 2018 Maddison et al., 2017, Le et al., 2018)

# Particle Filter



# Marginal Particle Filter



# Particle Filter

$$z_1^m \sim q(z_1), \quad w_1^m = p(z_1^m, x_1) / q(z_1^m)$$

For  $t = 2, \dots, T$  :

For  $m = 1, \dots, M$  :

Choose parent  $n \in \{1, \dots, M\}$  with  $\mathbb{P}(n) \propto w_{t-1}^n$

Sample  $z_t^m \sim q(z_t | z_{t-1}^n)$

Set  $w_t^m = p(z_t^m, x_t^m | z_{t-1}^n) / q(z_t^m | z_{t-1}^n)$

# Marginal Particle Filter

$$z_1^m \sim q(z_1), \quad w_1^m = p(z_1^m, x_1) / q(z_1^m)$$

For  $t = 2, \dots, T$  :

For  $m = 1, \dots, M$  :

~~Choose parent  $n \in \{1, \dots, M\}$  with  $\mathbb{P}(n) \propto w_{t-1}^n$~~

~~Sample  $z_t^m \sim q(z_t | z_{t-1}^n)$~~

~~Set  $w_t^m = p(z_t^m, x_t^m | z_{t-1}^n) / q(z_t^m | z_{t-1}^n)$~~

# Marginal Particle Filter

$$z_1^m \sim q(z_1), \quad w_1^m = p(z_1^m, x_1) / q(z_1^m)$$

For  $t = 2, \dots, T$  :

For  $m = 1, \dots, M$  :

Sample  $z_t^m$  from *mixture* of  $q(z_1 | z_{t-1}^1) \cdots q(z_1 | z_{t-1}^N)$   
with weights  $w_{t-1}^1 \cdots w_{t-1}^N$

~~Set  $w_t^m = p(z_t^m, x_t^m | z_{t-1}^n) / q(z_t^m | z_{t-1}^n)$~~

# Marginal Particle Filter

$$z_1^m \sim q(z_1), \quad w_1^m = p(z_1^m, x_1) / q(z_1^m)$$

For  $t = 2, \dots, T$  :

For  $m = 1, \dots, M$  :

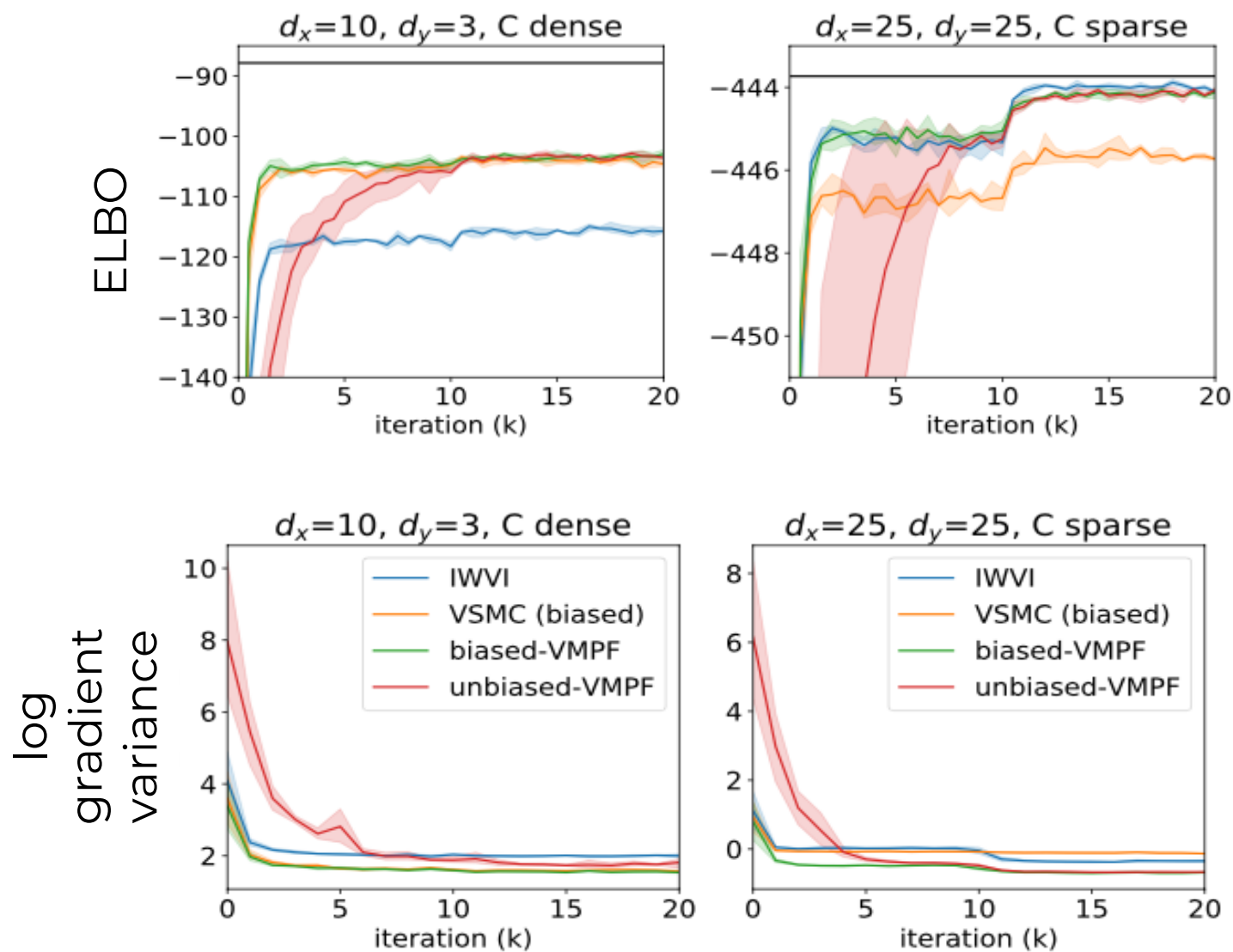
Sample  $z_t^m$  from *mixture* of  $q(z_1 | z_{t-1}^1) \cdots q(z_1 | z_{t-1}^N)$   
with weights  $w_{t-1}^1 \cdots w_{t-1}^N$

$$\text{Set } w_t^m = \frac{\sum_{n=1}^M w_{t-1}^n p(z_t^m, x_t^m | z_{t-1}^n)}{\sum_{n=1}^M w_{t-1}^n q(z_t^m | z_{t-1}^n)}$$

$$R_M := \prod_{t=1}^T \frac{1}{M} \sum_{m=1}^M w_t^m \quad \mathbb{E} R_M = p(x)$$

Maximize  $\mathbb{E} \log R_M$ , approximate  $p(z_T | x)$  (Lai, D., Sheldon, 2022)

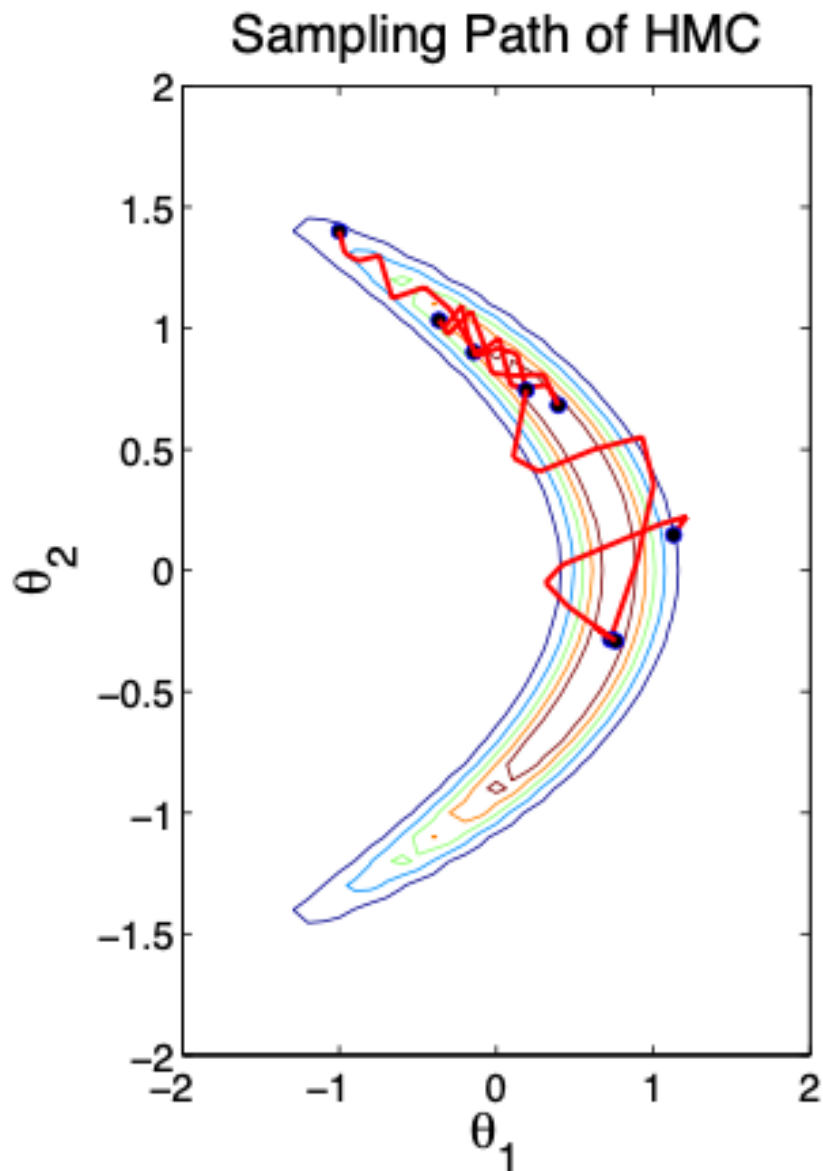
# Better... sometimes



|      |                                                 |
|------|-------------------------------------------------|
| IWVI | Importance Weighting                            |
| VSMC | (recent method)                                 |
| VMPF | Variational Marginal Particle Filter (proposed) |

# Variational Annealing

# MCMC



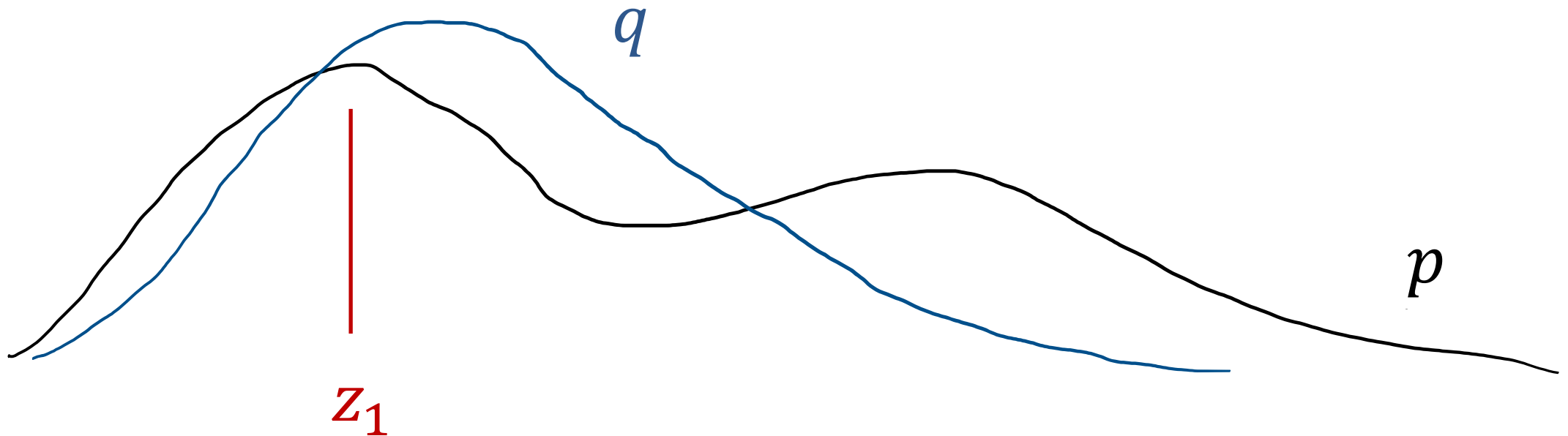
Idea: Find  $R$  such that  $\mathbb{E}R = p(x)$ ,  
optimize  $\mathbb{E} \log R_M$

1. Initial dist, mass matrix, step size, annealing schedule
2. Use MCMC to support maximum-likelihood learning

(Figure from Lan et al., 2012)

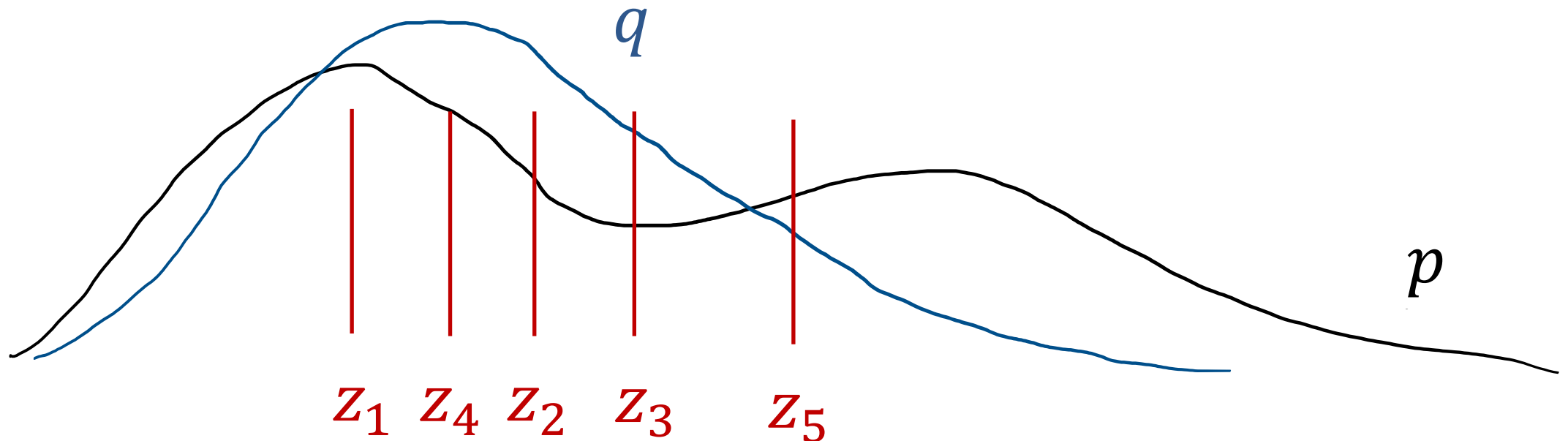
# MCMC

1. Draw  $z_1 \sim q(z)$
2. For  $k = 1, 2, \dots, K - 1$ , sample  $z_{k+1} \sim T(\cdot | z_k)$  where  $T$  is one iteration of MCMC



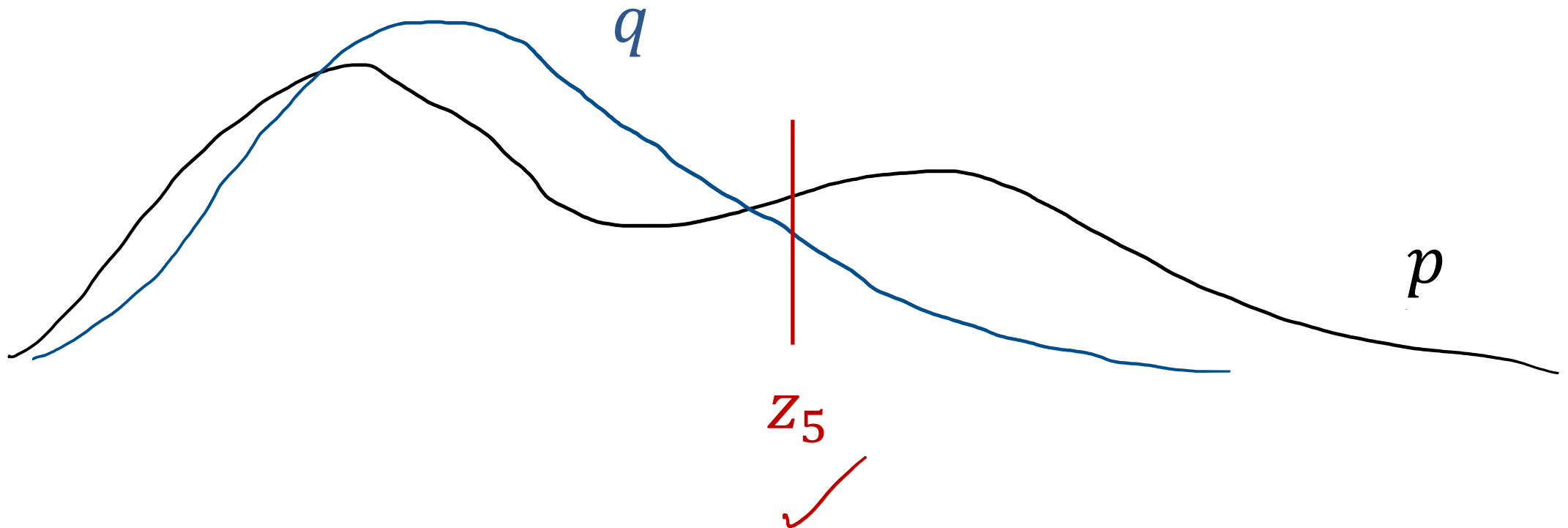
# MCMC

1. Draw  $z_1 \sim q(z)$
2. For  $k = 1, 2, \dots, K - 1$ , sample  $z_{k+1} \sim T(\cdot | z_k)$  where  $T$  is one iteration of MCMC



# MCMC

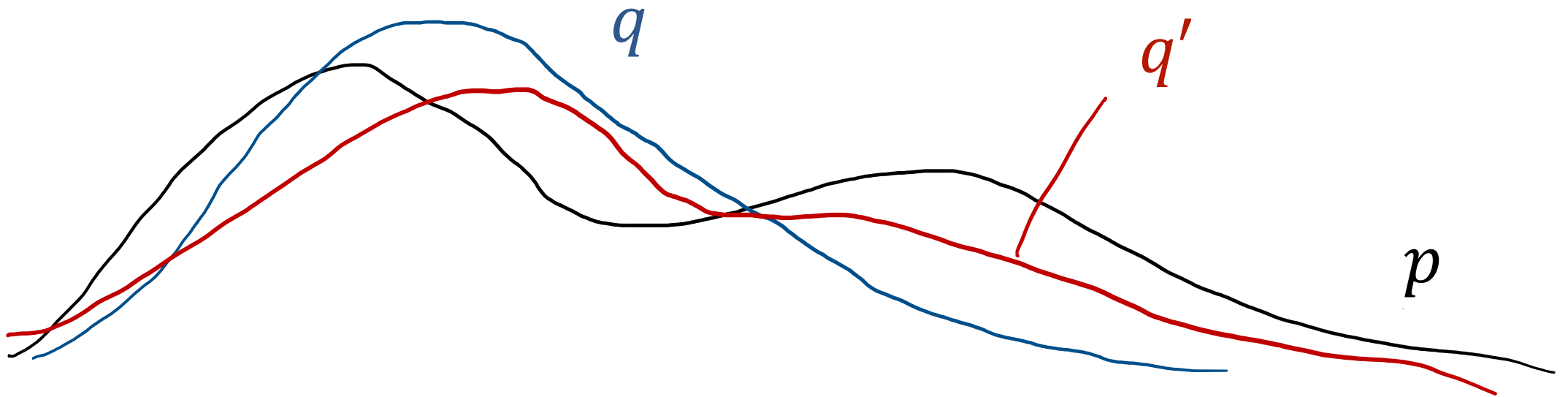
1. Draw  $z_1 \sim q(z)$
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# MCMC

1. Draw  $z_1 \sim q(z)$
2. For  $k = 1, 2, \dots, K - 1$ , sample  $z_{k+1} \sim T(\cdot | z_k)$  where  $T$  is one iteration of MCMC

$q'$



# Augmentation

Problem: Intractable to compute  $q'(z_K)$

Instead: Use full trace  $q'(z_1, \dots, z_K)$

- Augment  $p$  into  $p'(z_1, \dots, z_K, x)$
- Solve  $\min_{q \in \text{Family}} \text{KL}(q'(z_1, \dots, z_K) || p'(z_1, \dots, z_K | x))$

# Bridging the Gap

$$q'(z_1, \dots, z_K) = \overset{\text{learned starting dist}}{q(z_1)} \prod_{k=1}^{K-1} \overset{\text{one iteration of MCMC}}{q(z_{k+1}|z_k)}$$

$$p'(z_1, \dots, z_K, x) = \underset{\text{original target}}{p(z_K, x)} \prod_{k=1}^{K-1} \underset{\text{learned inverse dynamics}}{p(z_k|z_{k+1})}$$

**Good:** Can optimize “end to end” (dropping accept/reject)

**Bad:** Learning inverse dynamics is hard.

# Annealed Importance Sampling

$$q'(z_1, \dots, z_K) = \overset{\text{learned starting dist}}{q(z_1)} \prod_{k=1}^{K-1} \overset{\text{one iteration of MCMC on } \pi_m}{q(z_{k+1}|z_k)}$$
$$p'(z_1, \dots, z_K, x) = \underset{\text{original target}}{p(z_K, x)} \prod_{k=1}^{K-1} \underset{\text{"reversal" of } q(z_{m+1}|z_m) \text{ w.r.t. } \pi_m}{p(z_k|z_{k+1})}$$

Bridging densities  $\pi_1 \cdots \pi_K$  go from  $q(z_1)$  to  $p(z_K|x)$ .

**Good:**  $p'(z_1, \dots, z_K, x)/q'(z_1, \dots, z_K)$  has simple closed form.

**Good:** No need to learn inverse dynamics.

**Bad:** Must use accept/reject steps. Hard to optimize.

(Jarzynski 1997, Neal 2001)

# Uncorrected Hamiltonian Annealing

$$q'(z_1, \dots, z_K) = \overset{\text{learned starting dist}}{q(z_1)} \prod_{k=1}^{K-1} \overset{\text{one iteration of HMC on } \pi_m \text{ with NO CORRECTION}}{q(z_{k+1}|z_k)}$$

$$p'(z_1, \dots, z_K, x) = \overset{\text{original target}}{p(z_K, x)} \prod_{k=1}^{K-1} \overset{\text{"run HMC backwards" for one iteration}}{p(z_k|z_{k+1})}$$

Bridging densities  $\pi_1 \dots \pi_K$  go from  $q(z_1)$  to  $p(z_K|x)$ .

**Good:**  $p'(z_1, \dots, z_K, x)/q'(z_1, \dots, z_K)$  has simple closed form.

**Good:** No need to learn inverse dynamics.

**Good:** Fully differentiable, can optimize “end to end”.

(Geffner & D. 2021, Zhang et al. 2021)

# Algorithm

Sample state  $z_1 \sim q$  and momentum  $\rho_1 \sim S$ .

Initialize ELBO estimator as  $\mathcal{L} \leftarrow -\log q(z_1)$ .

For  $k = 1, 2, \dots, K - 1$ :

    Sample new momentum  $\rho'$ .

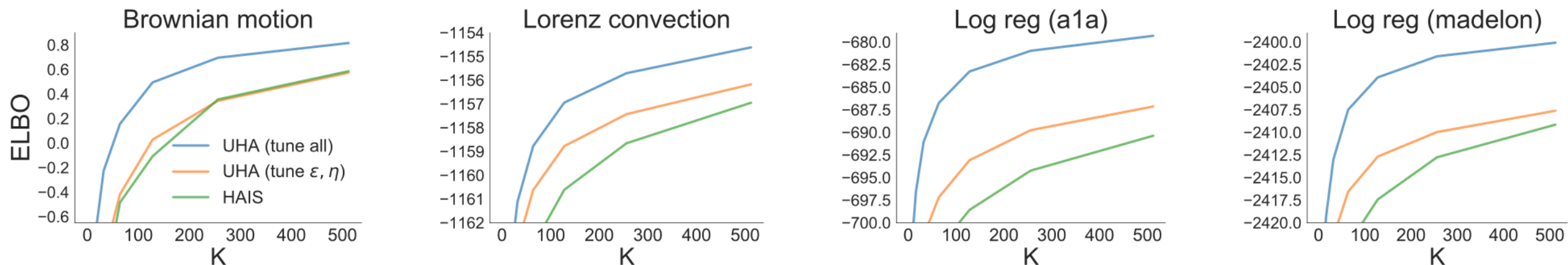
$(z_{k+1}, \rho_{k+1}) \leftarrow \text{HMC with target } \pi_k \text{ and starting point } (z_k, \rho')$ .

    Update estimator as  $\mathcal{L} \leftarrow \mathcal{L} + \log S(\rho_{k+1}) - \log S(\rho')$

Update estimator as  $\mathcal{L} \leftarrow \mathcal{L} + \log p(z_K, x)$

Can compute gradients for initial distribution  $q$ , bridging schedule, HMC hyperparameters, etc.

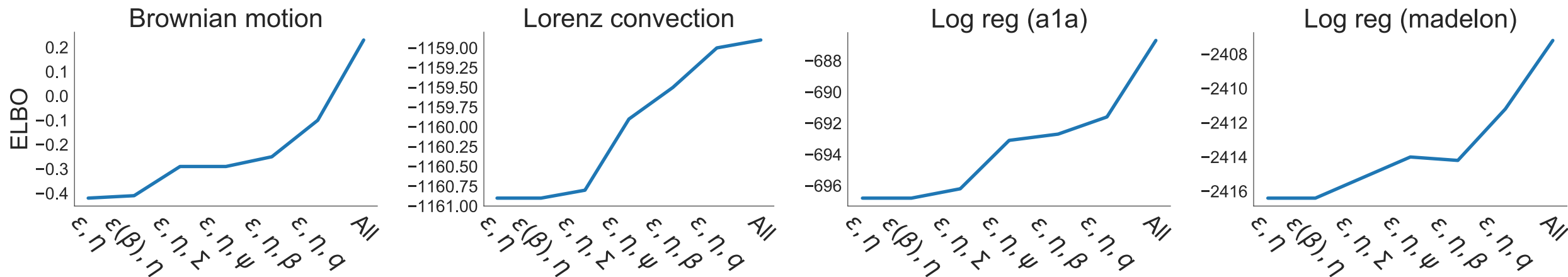
# Tuning stuff is good



$\epsilon$  – step size of HMC dynamics  
 $\eta$  – damping coefficient  
 $\Sigma$  – moment covariance  
 $\beta$  – temperature schedule  
 $\psi$  – “full rank” temperature schedule  
 $q$  – initial distribution

(K = total # of evaluations of  $p(z, x)$  per ELBO est.)

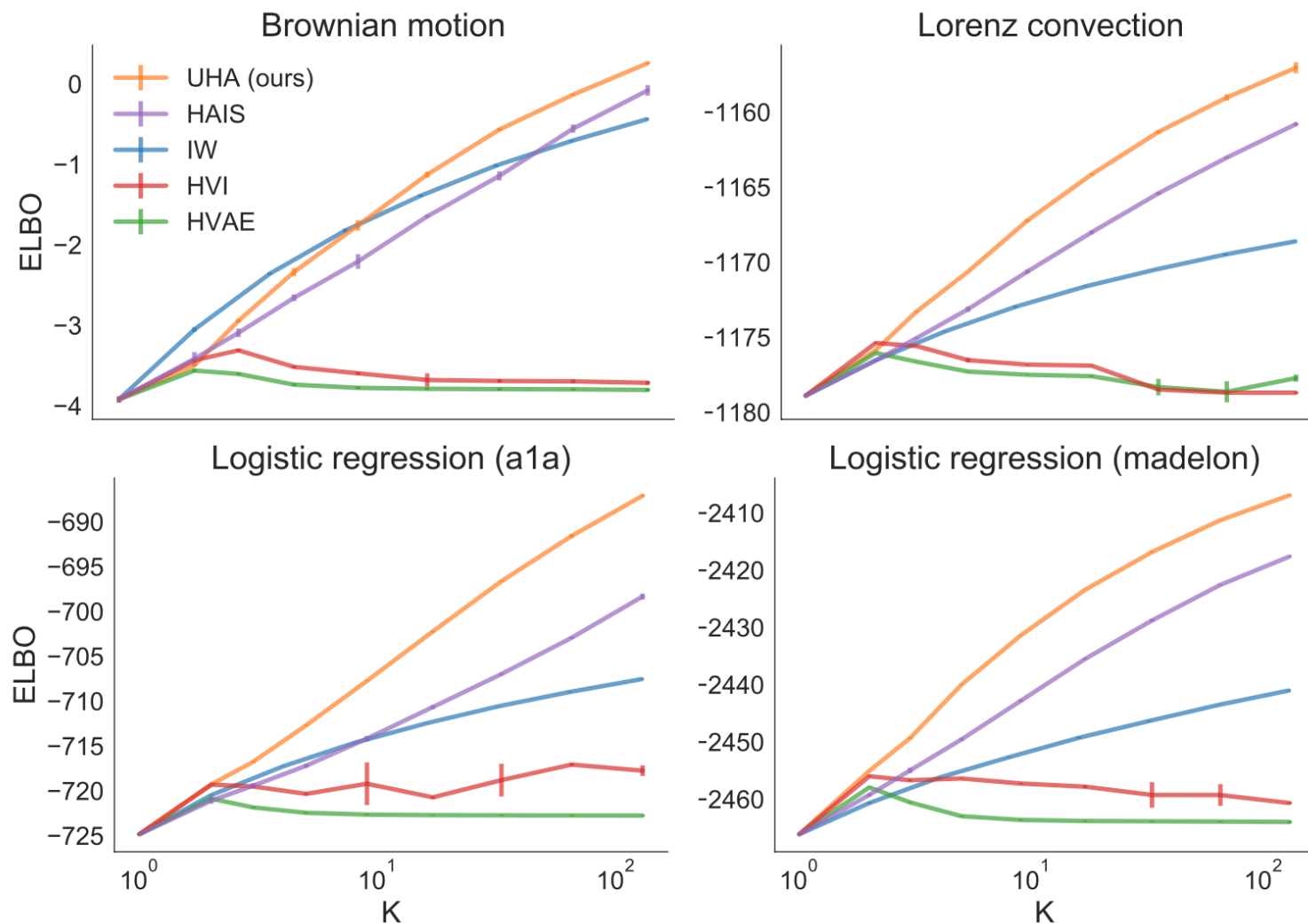
# Tuning more stuff is good



(K = 64)

$\epsilon$  – step size of HMC dynamics  
 $\eta$  – damping coefficient  
 $\Sigma$  – moment covariance  
 $\beta$  – temperature schedule  
 $\psi$  – “full rank” temperature schedule  
 $q$  – initial distribution

# Compares well to baselines



|      |                                                 |
|------|-------------------------------------------------|
| UHA  | Our algorithm                                   |
| HAIS | Annealed Importance Sampling using HMC dynamics |
| IW   | Importance Weighting                            |
| HVI  | “Bridging the gap” using HMC dynamics           |
| HVAE | (Recent algorithm)                              |

# VAE training

ELBO on test set

|         |     | $K = 1$ | $K = 8$ | $K = 16$ | $K = 32$ | $K = 64$ |
|---------|-----|---------|---------|----------|----------|----------|
| mnist   | UHA | -93.4   | -89.8   | -88.8    | -88.1    | -87.6    |
|         | IW  | -93.4   | -90.5   | -89.9    | -89.4    | -89.0    |
| letters | UHA | -137.9  | -133.5  | -132.3   | -131.5   | -130.9   |
|         | IW  | -137.9  | -134.6  | -133.9   | -133.2   | -132.7   |
| kmnist  | UHA | -184.2  | -176.6  | -174.6   | -173.2   | -171.6   |
|         | IW  | -184.2  | -179.7  | -178.7   | -177.8   | -177.0   |

log-likelihood on test set

|         |     | $K = 1$ | $K = 8$ | $K = 16$ | $K = 32$ | $K = 64$ |
|---------|-----|---------|---------|----------|----------|----------|
| mnist   | UHA | -88.5   | -87.5   | -87.2    | -87.0    | -86.9    |
|         | IW  | -88.5   | -87.6   | -87.5    | -87.3    | -87.2    |
| letters | UHA | -131.9  | -130.7  | -130.3   | -130.1   | -129.9   |
|         | IW  | -131.9  | -130.9  | -130.7   | -130.6   | -130.4   |
| kmnist  | UHA | -174.3  | -172.2  | -171.6   | -171.2   | -170.2   |
|         | IW  | -174.3  | -173.0  | -172.6   | -172.4   | -172.2   |

# Thank you!



Dan Sheldon



Jinlin Lai



Tomas Geffner

## **Publications**

*Importance Weighted Variational Inference*, NeurIPS, 2018

*Divide and Couple: Using Monte Carlo Variational Objectives for Posterior Approximation*, NeurIPS 2018

*Variational Marginal Particle Filters*, AISTATS 2022

*MCMC Variational Inference via Uncorrected Hamiltonian Annealing*, NeurIPS 2021