THE SAMPLE COMPLEXITY OF TOEPLITZ COVARIANCE ESTIMATION

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Joint with Yonina Eldar, Jerry Li, and Christopher Musco.
Covariance Estimation Problem. Consider positive semidefinite matrix $T \in \mathbb{R}^{d \times d}$ and distribution $\mathcal{D}$ over $d$-dimensional vectors with covariance $\mathbb{E}_{x \sim \mathcal{D}}[xx^T] = T$ (i.e., $T_{j,k}$ is the covariance between $x_j$ and $x_k$).
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Given independent samples $x^{(1)}, \ldots, x^{(n)} \sim \mathcal{D}$, return $\tilde{T}$ with:

$$\|T - \tilde{T}\|_2 \leq \varepsilon \|T\|_2.$$
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- Seems to be interesting even beyond Toeplitz covariance matrices, but not well studied.
EXAMPLE: DIRECTION OF ARRIVAL ESTIMATION

narrowband signal:
\[ s(t) = a(t) \cdot \cos(ft) \]
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\( x_1^{(j)} \quad \ldots \quad x_d^{(j)} \)
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Vector sample complexity:
Estimation time (# snapshots).

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Number of active receivers.

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\begin{align*}
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\text{With delay,} & \quad \mathbb{E}[x_k^{(j)} \cdot x_{\ell}^{(j)}] \approx \mathbb{E}[a(t)^2] \cdot \cos(f \Delta_{k,\ell}) \\
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**OUR CONTRIBUTIONS**

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- Show that sparse ruler methods give sublinear total sample complexity when $T$ is low-rank (e.g., DOA with $k \ll d$ senders).
Current state: Many algorithms for Toeplitz covariance estimation, but few formal results on sample complexities/tradeoffs.

Our contributions:

- Give non-asymptotic sample complexity bounds by analyzing classic algorithms, including those with sublinear entry sample complexity based on sparse ruler measurements.
- Show that sparse ruler methods give sublinear total sample complexity when $T$ is low-rank (e.g., DOA with $k \ll d$ senders).
- Develop improved algorithms in the low-rank setting using techniques from matrix sketching, leverage score-based sampling, and sparse Fourier transforms. Resemble popular ‘subspace methods’ such as MUSIC and ESPRIT.
BROADER AGENDA

Build connections between theoretical computer science and signal processing.
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- Leverage score/effective resistance sampling, sparse Fourier transforms $\iff$ sub-Nyquist sampling, Chebyshev interpolation, active sampling for Gaussian process regression
- Column-based matrix approximation, combinatorial sparsification $\iff$ nonlinear function approximation, Fourier-sparse approximations
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Apply tools from TCS to tackle fundamental signal processing problems. A *Universal Sampling Method for Reconstructing Signals with Simple Fourier Transforms* [AKMMVZ STOC ‘19]
For today, consider algorithms that sample $x^{(1)}, \ldots, x^{(n)} \sim \mathcal{D}$ with covariance $T$, read a fixed subset of entries $R \subseteq [d]$ from each $x^{(j)}$, and approximate $T$ using $x^{(1)}_R, \ldots, x^{(n)}_R \in \mathbb{R}^{|R|}$. How small can $R$ be? I.e., what is the minimal entry sample complexity of such an algorithm? For general (non-Toeplitz) $T$, require $|R| = d$. 

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\quad\text{vs.}\quad
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\] To notice correlation between $x^{(j)}$ and $x^{(k)}$ must read both.
For today, consider algorithms that sample $x^{(1)}, \ldots, x^{(n)} \sim \mathcal{D}$ with covariance $T$, read a fixed subset of entries $R \subseteq [d]$ from each $x^{(j)}$, and approximate $T$ using $x^{(1)}_R, \ldots, x^{(n)}_R \in \mathbb{R}^{|R|}$.

Entry sample complexity: $|R|$. Total sample complexity: $|R| \cdot n$. 
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For general (non-Toeplitz) $T$, require $|R| = d$.

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{vs.} \quad T_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

To notice correlation between $x_j$ and $x_k$ must read both.
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$$T = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{d-2} & a_{d-1} \\ a_1 & a_0 & a_1 & \cdots & \cdots & a_{d-2} \\ a_2 & a_1 & a_0 & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{d-2} & \cdots & \cdots & \cdots & a_1 & \vdots \\ a_{d-1} & a_{d-2} & \cdots & \cdots & a_1 & a_0 \end{bmatrix}$$
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\end{bmatrix}
\]

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\cdot a_1 = \mathbb{E}[x_2 \cdot x_3] = \mathbb{E}[x_d \cdot x_{d-1}].
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\[ a_1 = \mathbb{E}[x_2 \cdot x_3] = \mathbb{E}[x_d \cdot x_{d-1}]. \]

Will see that we can achieve $|R| = O(\sqrt{d})$. 
Definition (Ruler) A subset $R \subseteq [d]$ is a ruler if for every distance $s \in \{0, \ldots, d - 1\}$, there exist $j, k \in R$ with $j - k = s$. 
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E.g., for $d = 10$, $R = \{1, 2, 5, 8, 10\}$ is a ruler.
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**Claim** For any $d$ there exists a sparse ruler $R$ with $|R| = 2\sqrt{d}$

- Suffices to take $R = [1, 2, \ldots, \sqrt{d}] \cup [2\sqrt{d}, 3\sqrt{d}, \ldots, d]$. 

![Ruler diagram](image-url)
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If $R$ is a ruler, for each $s \in \{0, \ldots, d - 1\}$, there is at least one $k, \ell \in R$ with $|k - \ell| = s$ and thus with covariance

\[ \mathbb{E}[x_{k}^{(j)} \cdot x_{\ell}^{(j)}] = a_s. \]
**Sparse Ruler Based Estimation**

\[
T = \begin{bmatrix}
  a_0 & a_1 & a_2 & \cdots & a_{d-2} & a_{d-1} \\
  a_1 & a_0 & a_1 & \cdots & \cdots & a_{d-2} \\
  a_2 & a_1 & a_0 & \cdots & \cdots & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_{d-2} & \cdots & \cdots & \cdots & a_1 & \vdots \\
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\end{bmatrix}
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- If \( R \) is a ruler, for each \( s \in \{0, \ldots, d-1\} \), there is at least one \( k, \ell \in R \) with \( |k - \ell| = s \) and thus with covariance

  \[
  \mathbb{E}[x^{(j)}_k \cdot x^{(j)}_\ell] = a_s.
  \]

- Get at least one independent sample of \( a_s \) from every \( x^{(j)}_R \).
If $R$ is a ruler, for each $s \in \{0, \ldots, d - 1\}$, there is at least one $k, \ell \in R$ with $|k - \ell| = s$ and thus with covariance

$$\mathbb{E}[x_k^{(j)} \cdot x_\ell^{(j)}] = a_s.$$ 

Get at least one independent sample of $a_s$ from every $x_R^{(j)}$. With enough samples $n$ from $\mathcal{D}$, will converge on an estimate of each $a_s$ and so of the full matrix $T$. 
How many vector samples do we need? What do we pay for the optimal entry sample complexity of sparse rulers?
How many vector samples do we need? What do we pay for the optimal entry sample complexity of sparse rulers?

- How does the total sample complexity compare to methods that read every entry of each \( x^{(j)} \), e.g., estimating \( T \) with the empirical covariance \( \hat{T} = \frac{1}{n} \sum_j x^{(j)} x^{(j)^T} \).
Let $\mathcal{D} = \mathcal{N}(0, T)$ be a $d$-dimensional Gaussian with $a_0 = 1$. 
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- For $n = O\left(\frac{\log d}{\varepsilon^2}\right)$ all estimates of $a_s$ give error $|\varepsilon_s| \leq \varepsilon$. 

In the worst case, $\|\tilde{T}T\|_2 = d$ but if $\varepsilon_s$ were independent, $\|\tilde{T}T\|_2 = p_d$ [Meckes '07].

Setting $\varepsilon' = p_d$, $n = ~O\left(\frac{d^2}{\varepsilon^2}\right)$ would give $\|\tilde{T}T\|_2$. 

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\[
\tilde{T} = \begin{bmatrix}
a_0 + \varepsilon_0 & a_1 + \varepsilon_1 & a_2 + \varepsilon_2 & \cdots & a_{d-2} + \varepsilon_{d-2} & a_{d-1} + \varepsilon_{d-1} \\
a_1 + \varepsilon_1 & a_0 + \varepsilon_0 & a_1 + \varepsilon_1 & \cdots & \cdots & a_{d-2} + \varepsilon_{d-2} \\
a_2 + \varepsilon_2 & a_1 + \varepsilon_1 & a_0 + \varepsilon_0 & \cdots & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\
a_{d-2} + \varepsilon_{d-2} & \cdots & \cdots & \cdots & \cdots & a_1 + \varepsilon_1 \\
a_{d-1} + \varepsilon_{d-1} & a_{d-2} + \varepsilon_{d-2} & \cdots & \cdots & a_1 + \varepsilon_1 & a_0 + \varepsilon_0
\end{bmatrix}
\]
Let $\mathcal{D} = \mathcal{N}(0, T)$ be a $d$-dimensional Gaussian with $a_0 = 1$.

- For $n = O\left(\frac{\log d}{\varepsilon^2}\right)$ all estimates of $a_s$ give error $|\varepsilon_s| \leq \varepsilon$.

\[
\tilde{T} - T = \begin{bmatrix}
\varepsilon_0 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_{d-2} & \varepsilon_{d-1} \\
\varepsilon_1 & \varepsilon_0 & \varepsilon_1 & \cdots & \cdots & \varepsilon_{d-2} \\
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\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
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\end{bmatrix}
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- In the worst case, $\|\tilde{T} - T\|_2 = \varepsilon d$ but if $\varepsilon_s$ were independent, $\|\tilde{T} - T\|_2 \leq \varepsilon \sqrt{d}$ [Meckes ‘07].
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\end{bmatrix}$$

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- Setting $\varepsilon' = \varepsilon / \sqrt{d}$, $n = \tilde{O}\left(\frac{d}{\varepsilon^2}\right)$ would give
  $$\|\tilde{T} - T\|_2 \leq \varepsilon \leq \varepsilon\|T\|_2.$$
Let $\mathcal{D} = \mathcal{N}(0, T)$ be a $d$-dimensional Gaussian with $a_0 = 1$.

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- Setting $\varepsilon' = \varepsilon / \sqrt{d}$, $n = \tilde{O}\left(\frac{d}{\varepsilon^2}\right)$ would give $\|\tilde{T} - T\|_2 \leq \varepsilon \leq \varepsilon\|T\|_2$. 
Theorem. For any ruler $R \subset [d]$, covariance estimation with $R$ gives $\|\tilde{T} - T\|_2 \leq \varepsilon \|T\|_2$ with entry sample complexity $|R|$ and vector sample complexity $n = \tilde{O}\left(\frac{d}{\varepsilon^2}\right)$. 
Theorem. For any ruler $R \subset [d]$, covariance estimation with $R$ gives $\|\tilde{T} - T\|_2 \leq \varepsilon \|T\|_2$ with entry sample complexity $|R|$ and vector sample complexity $n = \tilde{O} \left( \frac{d}{\varepsilon^2} \right)$.

- Vector sample complexity matches the complexity of estimating an unstructured covariance with the empirical covariance but entry sample complexity can be $O(\sqrt{d})$ instead of $d$. 
**Theorem.** For any ruler $R \subset [d]$, covariance estimation with $R$ gives $\|\tilde{T} - T\|_2 \leq \varepsilon \|T\|_2$ with entry sample complexity $|R|$ and vector sample complexity $n = \tilde{O}\left(\frac{d}{\varepsilon^2}\right)$.

- Vector sample complexity matches the complexity of estimating an unstructured covariance with the empirical covariance but entry sample complexity can be $O(\sqrt{d})$ instead of $d$.
- Proof uses the Fourier structure of Toeplitz matrices.
Algorithm: For each $s \in \{0, 1\}$ approximate $a_s$ by average over the ruler $R$:

$$\tilde{a}_s = \frac{1}{n|R_s|} \sum_{j=1}^{n} \sum_{(k,\ell) \in R_s} x^{(j)}_k \cdot x^{(j)}_\ell$$

where $R_s = \{k, \ell \in R : |k - \ell| = s\}$.

Let $\tilde{T}$ be the Toeplitz matrix with $\tilde{a}_s$ on its $s^{th}$ diagonal.
Algorithm: For each $s \in \{0, 1\}$ approximate $a_s$ by average over the ruler $R$:

$$\tilde{a}_s = \frac{1}{n|R_s|} \sum_{j=1}^{n} \sum_{(k,\ell) \in R_s} x^{(j)}_k \cdot x^{(j)}_\ell$$

where $R_s = \{k, \ell \in R : |k - \ell| = s\}$.

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Let $\tilde{T}$ be the Toeplitz matrix with $\tilde{a}_s$ on its $s^{th}$ diagonal.

• Let $E = T - \tilde{T}$ and $e = a - \tilde{a}$. We want to bound $\|E\|_2$. 
Entry approximation to matrix approximation: Can bound $\|\tilde{T} - T\|_2 = \|E\|_2$ in terms of the Fourier transform of $e$. 

SPARSE RULER PROOF SKETCH
Entry approximation to matrix approximation: Can bound $\| \tilde{T} - T \|_2 = \| E \|_2$ in terms of the Fourier transform of $e$. 

$$
\| E \|_2 \| E \|_2 = \max_{f \in [0,1]} e^{2 sf} = \max_{f \in [0,1]} e^{2 sf} \sum_{s=0}^{\infty} e^{2 sf}.
$$
Entry approximation to matrix approximation: Can bound $\|\tilde{T} - T\|_2 = \|E\|_2$ in terms of the Fourier transform of $e$.

\[
\|E\|_2 \leq \|E_\infty\|_2 = \max_{f \in [0,1]} \hat{e} = \max_{f \in [0,1]} \sum_{s=0}^{d} e \cdot \sin(2\pi sf).
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(M_f)_{j,k} = \frac{\sin(2\pi sf)}{|R_s|}
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Can rewrite the Fourier transform as:

$$\|\tilde{T} - T\|_2 \leq \max_{f \in [0,1]} \sum_{s=0}^{d} [a_s - \tilde{a}_s] \cdot \sin(2\pi sf) = \max_{f \in [0,1]} \text{tr} \left( T_R - \tilde{T}_R, M_f \right)$$

where $T_R, \tilde{T}_R$ are the principal submatrices of $T$ and $\tilde{T}$ restricted to the indices in the ruler $R$. 
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\[ \|\tilde{T}_R - T_R\|_2 \leq \max_{f \in [0,1]} \text{tr} \left( T_R - \hat{T}_R, M_f \right) \]

**Concentration Bound:** (Hanson-Wright) For fixed \( f \), if \( n = \tilde{O}(1/\varepsilon^2) \) can bound the righthand side with high prob. by:

\[ \varepsilon \|T_R\|_2 \cdot \|M_f\|_F \leq \varepsilon \|T_R\|_2 \cdot \sqrt{d} \leq \varepsilon \|T\|_2 \cdot \sqrt{d} \]

since each entry of \( M_f = \frac{\sin(2\pi sf)}{|R_s|} \) for some s so \( \|M_f\|_F \leq \sqrt{d} \).
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- Setting \( \epsilon' = \epsilon/\sqrt{d} \) and union bounding over a net of \( f \) values gives our \( n = \tilde{O}(d/\epsilon^2) \) bound.
\[ \| \tilde{T}_R - T_R \|_2 \leq \max_{f \in [0,1]} \text{tr} \left( T_R - \hat{T}_R, M_f \right) \]

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- Setting \( \varepsilon' = \varepsilon / \sqrt{d} \) and union bounding over a net of \( f \) values gives our \( n = \tilde{O}(d/\varepsilon^2) \) bound.
- The more coverage \( R \) has (the larger the \( |R_s| \) is on average), the smaller \( \|M_f\|_F \) will be. Let’s us interpolate between minimal entry sample complexity and minimal vector sample complexity.
For $R = [d]$, coverage is maximal and $\|M_f\|_F = O(\sqrt{\log d})$, letting us achieve vector sample complexity $n = \tilde{O} \left( \frac{1}{\varepsilon^2} \right)$. 
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- Algorithm is equivalent to setting $T = \text{avg} \left( \frac{1}{n} \sum x^{(j)} x^{(j)^T} \right)$.
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- Algorithm is equivalent to setting $T = \text{avg} \left( \frac{1}{n} \sum x(i)x(i)^T \right)$.

- Improves on sample complexity of just using the empirical covariance by a $\tilde{O}(d)$ factor.
Total sample complexity is $O(\sqrt{d}) \cdot \tilde{O}(d) = \tilde{O}(d^{3/2})$ for sparse ruler vs. $d \cdot \tilde{O}(1) = \tilde{O}(d)$ for full sample estimation.
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- Prove bounds are tight when $T$ is the identity.
IS THERE ALWAYS A TRADEOFF?

- Total sample complexity is $\sim O(pd)$ for sparse ruler estimation vs. $\sim O(d)$ for full sample estimation.
- Sparse rulers give much better total sample complexity when $T$ is (approximately) low-rank.
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• Sparse rulers give much better total sample complexity when $T$ is (approximately) low-rank. **Can we explain this?**
Recall that we have with $n = \tilde{O}(1/\varepsilon^2)$ samples:

$$\|T - \tilde{T}\|_2 \leq \varepsilon \|T_R\|_2 \cdot \|M_f\|_F \leq \varepsilon \|TR\|_2 \sqrt{d} \leq \varepsilon \|T\|_2 \sqrt{d}.$$
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- If $T$ is the identity, $\|T\|_2 = \|T_R\|_2 = 1$. But this is ‘very’ full-rank.
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- Low-rank matrices cannot look like the identity – have significant off diagonal mass [MMW ‘19].
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- If \( T \) is the identity, \( \|T\|_2 = \|T_R\|_2 = 1 \). But this is ‘very’ full-rank.
- Low-rank matrices cannot look like the identity – have significant off diagonal mass [MMW ‘19].
- **Upshot:** Show \( \|T_R\|_2 \leq \frac{k}{\sqrt{d}} \|T\|_2 \). Setting \( \varepsilon' = \varepsilon/k \) obtain total sample complexity \( \tilde{O} \left( \frac{\sqrt{dk^2}}{\varepsilon^2} \right) \).
Remainder of the talk: Will sketch a different approach to low-rank Toeplitz covariance estimation using sparse Fourier transform methods.
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- Connections between these two approaches.
**Vandermonde Decomposition:** Any rank-$k$ Toeplitz $T \in \mathbb{R}^{d \times d}$ can be written as $F_S D F_S$ where $F_S \in \mathbb{R}^{d \times k}$ is an ‘off-grid’ Fourier transform matrix with frequencies $f_1, \ldots, f_k$ and $D$ is a positive diagonal matrix.
Vandermonde Decomposition: Any rank-$k$ Toeplitz $T \in \mathbb{R}^{d \times d}$ can be written as $F_S D F_S$ where $F_S \in \mathbb{R}^{d \times k}$ is an ‘off-grid’ Fourier transform matrix with frequencies $f_1, \ldots, f_k$ and $D$ is a positive diagonal matrix.

- Any sample $x \sim \mathcal{N}(0, T)$ can be written as $F_S D^{1/2} g$ for $g \sim \mathcal{N}(0, I)$. $\mathbb{E}[xx^T] = F_S D^{1/2} \mathbb{E}[gg^T] D^{1/2} F_S^* = T.$
$x \sim \mathcal{N}(0, T) = F_s D^{1/2} g$ is a Fourier sparse function.
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\[ x = D_{11} \cdot g_1 + D_{22} \cdot g_2 + \cdots + D_{kk} \cdot g_k \]

- Can recover exactly e.g. via Prony’s sparse Fourier transform method by reading any $2k$ entries.
$x \sim \mathcal{N}(0, T) = F_sD^{1/2}g$ is a **Fourier sparse function**.

- Can recover exactly e.g. via Prony’s sparse Fourier transform method by reading any $2k$ entries.
- Take $n = \tilde{O}(1/\varepsilon^2)$ samples, recover each in full by reading $2k$ entries, and then apply our earlier result for full ruler $R = [d]$. Total sample complexity: $\tilde{O}(k/\varepsilon^2)$. 
What about when $T$ is close to, but not exactly rank-$k$?
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- Prony’s method totally fails in this case.
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**Step 1:** Prove that when $T$ is close to low-rank, there is some set of $k$ frequencies that approximately spans each $x^{(i)} \sim \mathcal{N}(0, T)$. 
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- Not as easy as it sounds.

**Step 2:** Use a robust sparse Fourier transform method to approximately recover \( x^{(1)}, \ldots, x^{(n)} \) and then estimate \( T \) from these samples.

- Well studied in TCS, especially in the case when \( f_1, \ldots, f_k \) are ‘on grid’ integer frequencies.
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- We give a proof via a column subset selection result (see e.g., Guruswami Sinop ‘12):
**Step 1:** Prove that when $T$ is close to low-rank, there is some set of $k$ frequencies that approximately spans each $x^{(i)} \sim \mathcal{N}(0, T)$.

- We give a proof via a column subset selection result (see e.g., Guruswami Sinop ‘12):

**Theorem:** Any $A \in \mathbb{R}^{n \times d}$, contains a subset of $O(k/\varepsilon)$ columns, $C$ such that:

$$\|A - P_C \cdot A\|_F^2 \leq (1 + \varepsilon) \min_{\text{rank } - k M} \|A - M\|_F^2.$$
$x^{(1)}, \ldots , x^{(n)} \sim \mathcal{N}(0, T)$ can be written as $X = F_S D^{1/2} G$ where columns of $G$ are distributed as $\mathcal{N}(0, I)$. 
$x^{(1)}, \ldots, x^{(n)} \sim \mathcal{N}(0, T)$ can be written as $X = FSD^{1/2}G$ where columns of $G$ are distributed as $\mathcal{N}(0, I)$.
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- Think of $G$ as a linear sketch that ensures $\tilde{F}SD^{1/2}G \approx FSD^{1/2}$ (formally a projection-cost preserving sketch [CEMMP ‘15]).
\( x^{(1)}, \ldots, x^{(n)} \sim \mathcal{N}(0, T) \) can be written as \( X = F S D^{1/2} G \) where columns of \( G \) are distributed as \( \mathcal{N}(0, I) \).

- Think of \( G \) as a linear sketch that ensures \( F S D^{1/2} G \approx F S D^{1/2} \) (formally a projection-cost preserving sketch [CEMMP ‘15]).
- Apply column subset selection result to \( F S D^{1/2} \).
$x^{(1)}, \ldots, x^{(n)} \sim \mathcal{N}(0, T)$ can be written as $X = F_S D^{1/2} G$ where columns of $G$ are distributed as $\mathcal{N}(0, I)$.

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\[ X = F_S D^{1/2} G \]
$x^{(1)}, \ldots, x^{(n)} \sim \mathcal{N}(0, T)$ can be written as $X = F SD^{1/2}G$ where columns of $G$ are distributed as $\mathcal{N}(0, I)$.

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\[ X \approx F SD^{1/2}G \]
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Step 2: Recover frequencies $f_1, \ldots, f_m$ and $Z \in \mathbb{C}^{m \times n}$ with $X \approx F_M \cdot Z$. Then estimate $T$ using this approximation.

- Find frequencies via brute force search over a net.
- At each step of the search, for a given $F_M$, we must find $Z$ that reconstructs $X$ as well as possible using these frequencies. How do we do this without reading all of $X$?
Want to find $Z$ satisfying the approximate regression guarantee: 

$$\|X - F_M Z\|_F^2 = O(1) \cdot \min_Y \|X - F_M Y\|_F^2.$$
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- Suffices to sample $\tilde{O}(k)$ rows by the leverage scores of $F_M$ and solve the regression problem just considering these rows.
Want to find $Z$ satisfying the approximate regression guarantee:

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- **Remark:** If $f_1, \ldots, f_m$ are ‘on-grid’ integers, the columns of $F_M$ are orthonormal and the leverage scores are all $k/n$
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- Suffices to sample $\tilde{O}(k)$ rows by the leverage scores of $F_M$ and solve the regression problem just considering these rows.

- **Remark:** If $f_1, \ldots, f_m$ are ‘on-grid’ integers, the columns of $F_M$ are orthonormal and the leverage scores are all $k/n \to$ RIP for subsampled Fourier matrices.
Leverage scores measure much large a function in the column span of $F_M$ can be at index $i$ (i.e., how important that index may be in the regression.)

$$\tau_i(F_M) = \max_y \frac{(F_My)_i^2}{\|F_My\|_2^2}.$$
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- Using that $F_My$ is a Fourier sparse function we can bound this quantity a priori, without any dependence on $F_M$. 

![Graphs showing leverage scores](image-url)
Extend bounds of [Chen Kane Price Song ‘16] to give explicit function upper bounding the leverage scores of any $F_M$:
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Since this distribution is universal, can sample one set of entries by these leverages scores, and find $X \approx F_M \cdot Z$ with high probability for any set of frequencies $f_1, \ldots, f_m$ in net.
1. Sample polynomial \( k \) indices \( R \) \( \{d\} \) according to the sparse Fourier leverage distribution (a random ‘ultra-sparse’ ruler).

2. For all \( f_1; : : : ; f_m \) in net \( N \): Compute approximate projection:

\[
Z = \arg \min \{ Z \} \in C_m \| \mathbf{X}_R(F_M) R Z \|_2^2
\]

3. Set \( \tilde{\mathbf{X}} = F_{\star} M_{\star} Z_{\star} \) to the best frequency-based approximation.

4. Return \( \tilde{T} = \text{avg}(\tilde{\mathbf{X}} \tilde{\mathbf{X}}^T) \).

Sample Complexity:

Gives \( \| T \|_2^2 \leq \| T \|_2^2 + f(T) k \) when \( X \) contains \( n = \tilde{O}(\text{poly}(k = \epsilon)) \) samples. Entry sample complexity \( \text{poly}(k = \epsilon) \), total sample complexity \( \tilde{O}(\text{poly}(k = \epsilon)) \).
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4. Return $\tilde{T} = \text{avg}(\tilde{X} \tilde{X}^T)$.

**Sample Complexity:** Gives $\| T - \tilde{T} \|_2 \leq \varepsilon \| T \|_2 + f(T - T_k)$ when $X$ contains $n = \tilde{O}(\text{poly}(k/\varepsilon))$ samples. Entry sample complexity poly($k/\varepsilon$), total sample complexity $\tilde{O}(\text{poly}(k/\varepsilon))$. 

---
Concrete.

- Runtime efficiency?
  - Can likely avoid exponential time net approach using off-grid sparse Fourier transform of [Chen Kane Price Song '16.]
- Convex optimization-based approaches and 'off-grid' RIP?
- Matrix sparse Fourier transform $X_F M Z$.
  - Connections to MUSIC, ESPRIT, etc.
- In process, maybe improve our sample complexity.
- 'Continuous' setting with sample access to a arbitrary positions of a signal with stationary covariance. (E.g., $x(1); \ldots; x(n)$ may be snapshots of this signal.)
- Sample complexity bounds and tradeoffs for applications like direction-of-arrival estimation, Doppler imaging.
Concrete.
Concrete.

• Runtime efficiency?
Concrete.

- Runtime efficiency?
  - Can likely avoid exponential time net approach using off-grid sparse Fourier transform of [Chen Kane Price Song ‘16.]
  - Convex optimization-based approaches and ‘off-grid’ RIP?
  - Matrix sparse Fourier transform $X \approx F_M \cdot Z$. Connections to MUSIC, ESPRIT, etc.
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  - Sample complexity bounds and tradeoffs for applications like direction-of-arrival estimation, Doppler imaging.
CONNECTIONS BETWEEN SAMPLING SCHEMES
Connections between sampling schemes

Fourier Sparse Leverage Scores

Optimal Sparse Ruler for $d=91$

Degree 40 Chebyshev Nodes
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  • Seem to have a lot more to understand.
Thanks! Questions?

Paper draft and slides available at cameronmusco.com