# COMPSCI 690RA: Randomized Algorithms and Probabilistic Data Analysis

Prof. Cameron Musco
University of Massachusetts Amherst. Spring 2022.
Lecture 2

# Logistics

- Reminder that there is a weekly quiz, released after class on Wednesday and due the next Tuesday 8pm.
- Problem Set 1 was released Monday. Due next Friday 2/11. Download from the course website.
- See Piazza for a post to organize homework groups.
- Reminder that we encourage you to post your questions publicly on Piazza – you will receive extra credit for this. And help your classmates!

# Talk This Week

Thursday at 4pm Talya Eden (BU, MIT) will be giving a Zoom talk on Sublinear-Time Graph Algorithms: Motif Counting and Uniform Sampling.

- This is a very cool line of work that heavily uses randomization.
- Link on CICS Events page.

`https://umass-amherst.zoom.us/j/94725490374?pwd=bGtsa0hjNGx5c1VyNnlGT21WbU5wQT09`

# Summary

Last Time:

$AB \stackrel{?}{=} C$

- Motivation behind randomized algorithms and some classic examples — polynomial identity testing, Freivald's algorithm.
- Complexity classes related to randomized algorithms – $P \subseteq ZPP \subseteq RP \subseteq BPP$.
- Probability review – linearity of expectation and variance.

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Today:

- Concentation bounds – Markov's and Chebyshev's inequalities.
- The union bound.
- Exponential concentration bounds – Chernoff and Bernstein
- Applications of tools to Quicksort analysis, coupon collecting, statistical estimation, random hashing.

# Application 1: Quicksort with Random Pivots

# Quicksort

`Quicksort`(X): where $X = (x_1, \ldots, x_n)$ is a list of numbers.

1. If $X$ is empty: return $X$.
2. Else: select pivot $p$ uniformly at random from $\{1, \ldots, n\}$.
3. Let $X_{lo} = \{i \in X : x_i < x_p\}$ and $X_{hi} = \{i \in X : x_i \geq x_p\}$ (requires $n - 1$ comparisons with $x_p$ to determine).
4. Return the concatenation of the lists [`Quicksort`($X_{lo}$), ($x_p$), `Quicksort`($X_{hi}$)].

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| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 4 | 5 | 2 | 8 | 1 | 3 | 6 | 9 | 7 | 0 |

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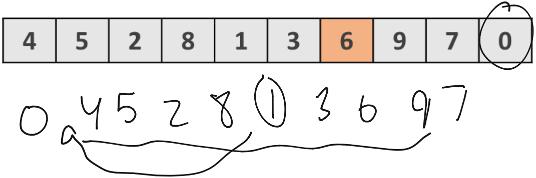

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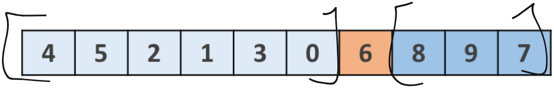

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What is the worst case running time of this algorithm?

$O(n^2)$

# Randomized Quicksort Analysis

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So we need to upper bound $\Pr[x_i, x_j \text{ are compared}]$.

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$[x_2 \quad x_1 \quad x_4 \quad x_3]$

- Assume without loss of generality that $x_1 \leq x_2 \leq \ldots \leq x_n$. This is just 'renaming' the elements of our list. Also recall that $i < j$.

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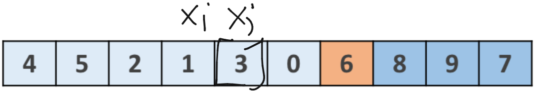

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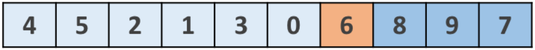

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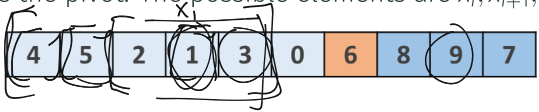


- $\Pr[x_i, x_j$ are compared$]$ is equal to the probability that either $x_i$ or $x_j$ are chosen as the splitting pivot from this list. Thus, $\Pr[x_i, x_j \text{ are compared}] = \frac{2}{j-i+1}$

# Randomized Quicksort Analysis

**So Far:** Expected number of comparisons is given as:

$$\mathbb{E}[T] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr[x_i, x_j \text{ are compared}].$$

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$\frac{2}{1}$ $\frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \dots \frac{2}{n-i+1}$

$2 + 3 + \dots, n-i+1$

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$$\leq \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} \leq 2 \cdot (n-1) \cdot \sum_{k=1}^{n} \frac{1}{k} = 2n \cdot H_n = O(n \log n).$$

$n^{th}$ Harmonic number

$H_n$

$2n\log n$

$n\log n$ is optimal for comparison based sorts

# Concentration Inequalities

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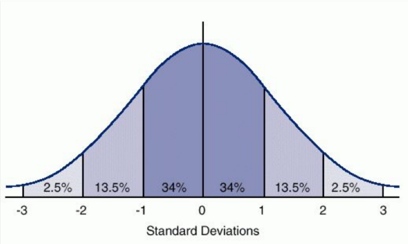
Concentration inequalities are bounds showing that a random variable lies close to it's expectation with good probability. Key tools in the analysis of randomized algorithms.



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non negativity

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$u \geq t$

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$$= t \cdot \Pr(X \geq t).$$

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Plugging in $t = \mathbb{E}[X] \cdot s$, $\Pr[X \geq s \cdot \mathbb{E}[X]] \leq 1/s$. The larger the deviation $s$, the smaller the probability.

# Markov's Inequality

ZPP ⊆ BPP

always right

**Think-Pair-Share:** You have a Las Vegas algorithm that solves some decision problem in expected running time $T$. Show how to turn this into a Monte-Carlo algorithm with worst case running time $3T$ and success probability $2/3$.

~~run algo 3 times using T steps each~~

time T+1

run algo. once for up to 3T steps. If terminates output answer else out FAIL

$\Pr(\text{correct}) = \Pr(\text{runtime} < 3T) \geq 2/3$

$\Pr(\text{runtime} > 3T) \leq \frac{E[\text{runtime}]}{3T} \leq \frac{1}{3}$

# Chebyshev's inequality

With a very simple twist, Markov's Inequality can be made much more powerful in many settings.

For any random variable X and any value $t > 0$:

$$\Pr(|X| \geq t) = \Pr(X^2 \geq t^2).$$

$$\leq \frac{\mathbb{E}[X^2]}{t^2} \quad \text{by Markov's}$$

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Plugging in the random variable $X - \mathbb{E}[X]$, gives the standard form of **Chebyshev's inequality:**

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathbb{E}[(X - \mathbb{E}[X])^2]}{t^2} = \frac{\mathrm{Var}(X)}{t^2}.$$

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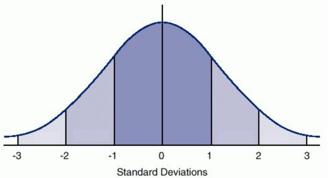
What is the probability that X falls $s$ standard deviations from it's mean?




$$\Pr(|X - \mathbb{E}[X]| \geq s \cdot \sqrt{\mathrm{Var}[X]}) \leq \frac{\mathrm{Var}[X]}{s^2 \cdot \mathrm{Var}[X]} = \frac{1}{s^2}.$$

# Application 2: Statistical Estimation + Law of Large Numbers

# Concentration of Sample Mean

**Theorem:** Let $X_1, \ldots, X_n$ be pairwise independent random variables with $\mathbb{E}[X_i] = \mu$ and $\text{Var}[X_i] = \sigma^2$. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_n$ be their sample average.

For any $\epsilon > 0$, $\Pr[|\bar{X} - \mu| \geq \epsilon\sigma] \leq \frac{1}{n\epsilon^2}$.

$\bar{X} \approx \mu$

samples $\uparrow$ prob $\downarrow$

$\varepsilon$ $\uparrow$ prob $\downarrow$

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- By linearity of expectation, $\mathbb{E}[\bar{X}] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[X_i] = \mu$.
- By linearity of variance, $\mathbb{E}[\bar{X}] = \frac{1}{n^2}\sum_{i=1}^{n} \mathrm{Var}[X_i] = \frac{\sigma^2}{n}$.

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- By linearity of variance, $\mathbb{E}[\bar{X}] = \frac{1}{n^2}\sum_{i=1}^{n}\text{Var}[X_i] = \frac{\sigma^2}{n}$.
- Plugging into Chebyshev's inequality:

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$n = \frac{1}{\varepsilon^2 \delta}$

$\delta > 0$

$\Pr(|\bar{X} - \mu| \geq \varepsilon\sigma) \leq \frac{1}{n\varepsilon^2} = \delta$

This is the weak law of large numbers.

$\varepsilon$ "error"

# Concentration of Sample Mean

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$\Pr[X_i = 1] = p$

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$\sum X_i$ is binomial

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**Upshot:** If we take $n = \frac{p}{\epsilon^2 \delta}$ samples, then with probability at least $1 - \delta$, $\bar{p}$ will be a $\pm\epsilon$ estimate to the true proportion $p$. A prototypical sublinear time algorithm.

# Application to Success Boosting

decision problem $n \cdot (1/\delta)$

**Think-Pair-Share:** You have a Monte-Carlo algorithm with worst case running time $T$ and success probability $2/3$. Show how to obtain, for any $\delta \in (0, 1)$, a Monte-Carlo algorithm with worse case running time $O(T/\delta)$ and success probability $1 - \delta$.

$T \cdot 100$ $\quad \delta = .01 \quad .99$

$X_i = 1$ if correct on rep $i$; $0$ if incorrect

repeat algo $n$ time

$$\mathbb{E}\left[\sum X_i\right] = \sum \mathbb{E} X_i = \frac{2}{3} \cdot n$$

$$\mathrm{Var}\left(\sum X_i\right) = \sum \mathrm{Var}(X_i) = n \cdot \frac{2}{3}\left(1 - \frac{2}{3}\right) = n \cdot \frac{2}{9}$$

If $\sum X_i \geq \frac{n}{2}$ then majority is correct

$$\Pr\left(\sum X_i \leq \frac{n}{2}\right) \leq \Pr\left(\left|\sum X_i - \mathbb{E}\sum X_i\right| \geq \frac{n}{6}\right) \leq \frac{n \frac{2}{9}}{n^2/6^2} = O\left(\frac{1}{n}\right)$$

# Application 3: Coupon Collecting

# Coupon Collector Problem

There is a set of $n$ unique coupons. At each step you draw a random coupon from this set. How many steps does it take you to collect all the coupons?

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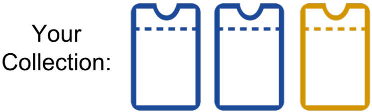

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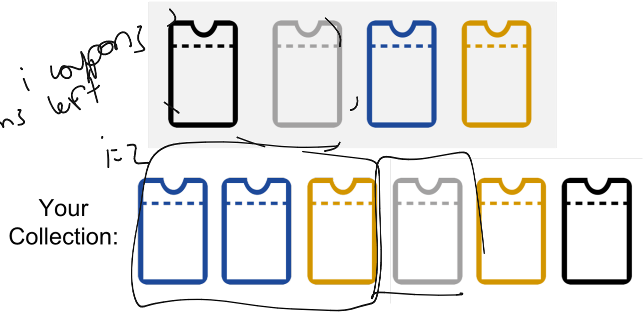


**Think-Pair-Share:** Say you have collected $i$ coupons so far. Let $T_{i+1}$ denote the number of draws needed to collect the $(i+1)^{st}$ coupon. What is $\mathbb{E}[T_i]$?

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- $\mathsf{T}_i$ is a geometric random variable with success probability $p_i = \frac{n-i}{n}$. I.e., $Pr[\mathsf{T}_i = j] = p_i(1-p_i)^{j-1}$.
- **Exercise:** verify that $\mathbb{E}[\mathsf{T}_i] = 1/p_i = \frac{n}{n-i}$.

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- By linearity of expectation, the expected number of draws to collect all the coupons is:

$$\mathbb{E}[T] = \sum_{i=0}^{n-1} \mathbb{E}[T_i] = \frac{n}{n} + \frac{n}{n-1} + \dots \frac{n}{n-(n-1)}$$

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$$= n \cdot H_n.$$

- By Markov's inequality, $\Pr[\mathrm{T} \geq cn \cdot H_n] \leq \frac{1}{c}$

# Quiz Question

Consider rolling a fair 6-sided dice, which takes a value in $\{1, 2, 3, 4, 5, 6\}$ each with probability 1/6. What is the expected number of rolls needed to see each odd number (i.e., see each of $\{1, 3, 5\}$) at least once?

$$\mathbb{E}[T] = \mathbb{E}[T_1] + \mathbb{E}[T_2] + \mathbb{E}[T_3]$$

$$= 2 + 3 + 6 = \boxed{11}$$

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- Putting these together:

$1, \frac{n-1}{n}, \frac{n-2}{n} \ldots$

$$\text{Var}[\mathsf{T}] = \sum_{i=0}^{n} \frac{1-p_i}{p_i^2} = \sum_{i=0}^{n} \frac{1}{p_i^2} - \sum_{i=0}^{n} \frac{1}{p_i}$$

$nH_n$

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$$\leq n^2 \cdot \frac{\pi^2}{6} - n \cdot H_n \leq n^2 \frac{\pi^2}{6}$$

$$\sum_{i=0}^{n} \frac{n^2}{(n-i)^2} = n^2\left(\frac{1}{1^2} + \frac{1}{2^2} + \ldots \frac{1}{n^2}\right) \quad \left(\ldots \leq \pi^2/6\right)$$

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c any number ≥ 0

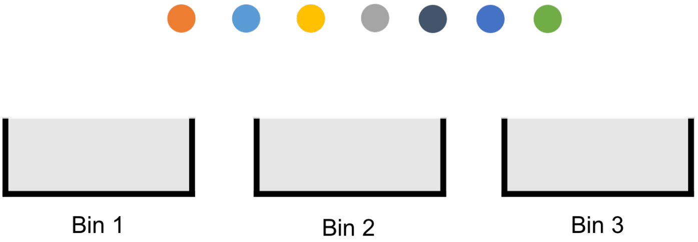
- Via Chebyshev's inequality, $\Pr[|\mathsf{T} - n \cdot H_n| \geq cn] \leq$ $\frac{\mathrm{Var}(T)}{c^2 n^2} = \frac{6}{c^2\pi^2}$ $\approx \frac{1}{c^2}$

$\frac{cn}{nH_n} \to 0$ as $n \to \infty$

# Application 4: Randomized Load Balancing and Hashing, and ‘Ball Into Bins’

# Balls Into Bins

I throw $m$ balls independently and uniformly at random into $n$ bins. What is the maximum number of balls any bin?



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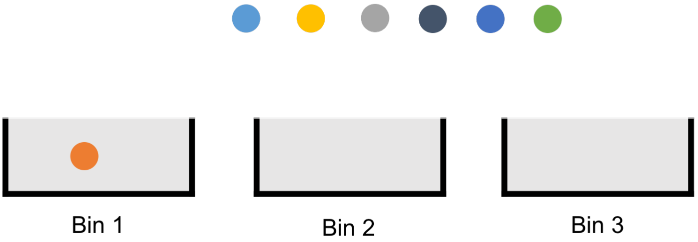

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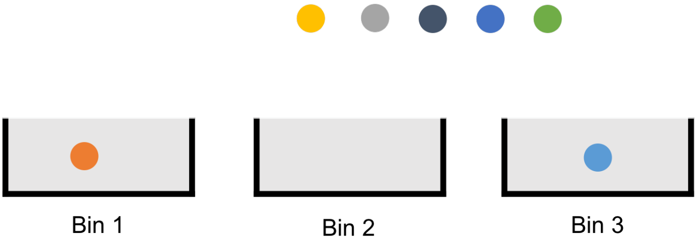

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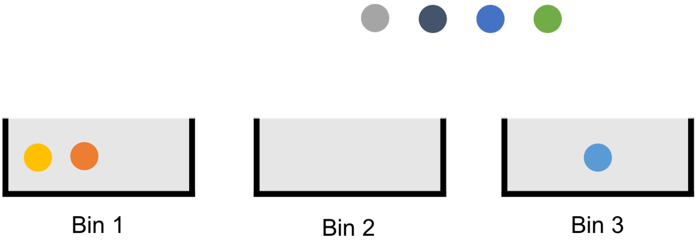

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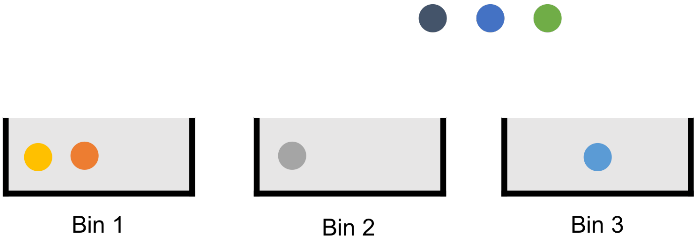

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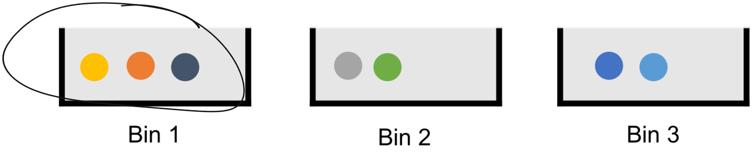
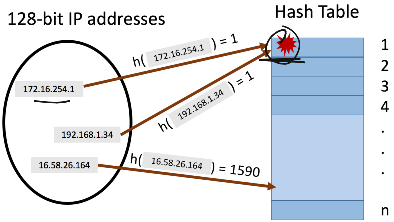


# Application: Hash Tables




- **hash function** $h : U \to [n]$ maps elements to indices of an array.
- Repeated elements in the same bucket are stored as a linked list – 'chaining'.
- Worse-case look up time is proportional to the maximum list length – i.e., the maximum number of 'balls' in a 'bin'.

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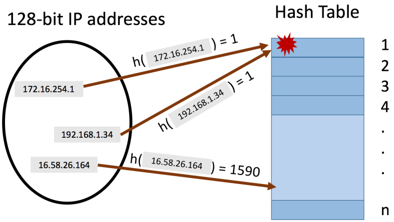


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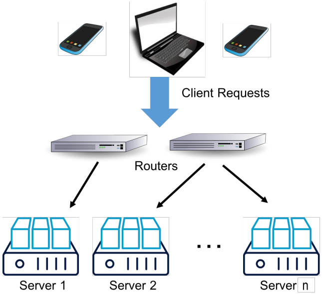
**Note:** A 'fully random hash function' maps items independently and uniformly at random to buckets. This is a theoretical idealization of practical hash functions.

# Application: Randomized Load Balancing




- $m$ requests are distributed randomly to $n$ servers. Want to bound the maximum number of requests that a single server must handle.
- Assignment is often is done via a random hash function so that repeated requests or related requests can be mapped to the same server, to take advantages of caching and other optimizations.

Let $\mathbf{b}_i$ be the number of balls landing in bin $i$. For $n$ balls into $n$ bins what is $\mathbb{E}[\mathbf{b}_i]$? $= \frac{m}{n}$

# Balls Into Bins Analysis

Let $\mathbf{b}_i$ be the number of balls landing in bin $i$. For $n$ balls into $m$ bins what is $\mathbb{E}[\mathbf{b}_i]$?

# balls in bin i

$$\Pr\left[\max_{i=1,\ldots,n} \mathbf{b}_i \geq k\right] = \Pr\left[\bigcup_{i=1}^{n} A_i\right],$$

where $A_i$ is the event that $\mathbf{b}_i \geq k$.

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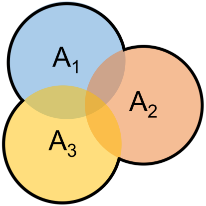
**Union Bound:** For any random events $A_1, A_2, \ldots, A_n$,

$$\Pr\left(A_1 \cup A_2 \cup \ldots \cup A_n\right) \leq \Pr(A_1) + \Pr(A_2) + \ldots + Pr(A_n).$$



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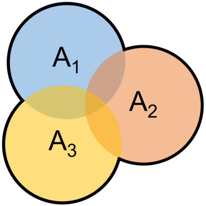


**Exercise:** Show that the union bound is a special case of Markov's inequality with indicator random variables.

# Balls Into Bins Direct Analysis

Let $\mathbf{b}_i$ be the number of balls landing in bin $i$. If we can prove that for any $i$, $\Pr[A_i] = \Pr[\mathbf{b}_i \geq k] \leq p$, then by the union bound:

p + p + p ...

$$\Pr\left[\max_{i=1,\ldots,n} \mathbf{b}_i \geq k\right] = \Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq n \cdot p.$$

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balls bins 10

**Claim 1:** Assume $m = n$. For $k \geq \frac{c \ln n}{\ln \ln n}$, $\Pr[\mathbf{b}_i \geq k] \leq \frac{1}{n^{c - o(1)}}$.

$\leq \frac{1}{n^{10 - o(1)}}$

$\Pr\left(\max b_i \geq \frac{c \ln n}{\ln\ln n}\right) \leq \frac{1}{n^{9 - o(1)}}$

$O\left(\frac{\ln n}{\ln \ln n}\right)$

# Balls Into Bins Direct Analysis

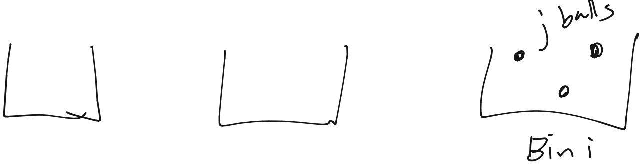
Let $\mathbf{b}_i$ be the number of balls landing in bin $i$. If we can prove that for any $i$, $\Pr[A_i] = \Pr[\mathbf{b}_i \geq k] \leq p$, then by the union bound:

$$\Pr\left[\max_{i=1,\ldots,n} \mathbf{b}_i \geq k\right] = \Pr\left[\bigcup_{i=1}^{n} A_i\right] \leq n \cdot p.$$

**Claim 1: Assume** $m = n$. For $k \geq \frac{c \ln n}{\ln \ln n}$, $\Pr[\mathbf{b}_i \geq k] \leq \frac{1}{n^{c-o(1)}}$.

- $\mathbf{b}_i$ is a binomial random variable with $n$ draws and success probability $1/n$.

$$\Pr[\mathbf{b}_i = j] = \binom{n}{j} \cdot \frac{1}{n^j} \cdot \left(1 - \frac{1}{n}\right)^{n-j}.$$



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- Summing over $j \geq k$ we have:

$$\Pr[\mathbf{b}_i \geq k] \leq \sum_{j \geq k} \left(\frac{e}{j}\right)^j = \left(\frac{e}{k}\right)^k \cdot \frac{1}{1 - e/k}.$$

$$\leq \sum_{j \geq k} \left(\frac{e}{k}\right)^j = \left(\frac{e}{k}\right)^k \cdot \sum_{i=0}^{\infty} \left(\frac{e}{k}\right)^i$$

# Balls Into Bins Direct Analysis

**We just showed:** When $n = m$ (i.e., $n$ balls into $n$ bins)

$$\Pr[\mathbf{b}_i \geq k] \leq \left(\frac{e}{k}\right)^k \cdot \frac{1}{1 - e/k}$$

For $k = \frac{c \ln n}{\ln \ln n}$ we have:

$$\Pr[\mathbf{b}_i \geq k] \leq \left(\frac{\ln \ln n}{\ln n}\right)^{\frac{c \ln n}{\ln \ln n}} \cdot \frac{1}{1 - (e \ln \ln n)/(c \ln n)}$$

$n \to \infty$

$0$

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$$\left[e^{\ln\ln\ln n - \ln\ln n}\right]^{\frac{c\ln n}{\ln\ln n}}$$

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$$e^{\left(\frac{c \ln\ln\ln n \cdot \ln n}{\ln\ln n}\right) - c\ln n}$$

$$e^{(o(1) - c)\ln n} = n^{o(1) - c}$$

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**Upshot:** By the union bound, For $k = c \frac{\ln n}{\ln \ln n}$ for sufficiently large $c$,

$$\Pr\left[\max_{i=1,\ldots,n} \mathbf{b}_i \geq k\right] \leq n \cdot \frac{1}{n^{c - o(1)}} = \frac{1}{n^{c - 1 - o(1)}}.$$

When throwing $n$ balls in to $n$ bins, with very high probability the maximum number of balls in a bin will be $O\left(\frac{\ln n}{\ln \ln n}\right)$.

# Balls Into Bins Via Chebyshev's Inequality

n bins, n balls

In our balls into bins analysis we directly bound $\Pr[\mathbf{b}_i \geq k] \leq \left(\frac{e}{k}\right)^k \cdot \frac{1}{1-e/k}$.

**Think Pair Share:** Give an upper bound on this probability using Chebyshev's inequality. Hint: write $\mathbf{b}_i$ as a sum of $n$ indicator random variables and compute $\mathrm{Var}[\mathbf{b}_i]$.

$b_i = \sum x_{ij}$

$\mathrm{Var}[b_i] = \sum \mathrm{Var}(x_{ij}) \leq n \cdot \frac{1}{n} \leq 1$

$\frac{1}{n}(1-\frac{1}{n})$

$\Pr(b_i \geq k) \leq \frac{\mathbb{E} b_i^2}{k^2} \leq \frac{2}{k^2}$

$\mathrm{Var}[b_i] = \mathbb{E} b_i^2 - (\mathbb{E} b_i)^2$

$\leq 1$ $\qquad \leq 2$ $\qquad 1$

$\mathbb{E} b_i^2 \leq 2$

# Balls Into Bins Via Chebyshev's Inequality

**By Chebyshev's Inequality:** $\Pr[\mathbf{b}_i \geq k] \leq \frac{2}{k^2}$.

Setting $k = c\sqrt{n}$, $\Pr\left[\mathbf{b}_i \geq c\sqrt{n}\right] \leq \frac{2}{c^2 n}$. So via a union bound:

$$\Pr\left[\max_{i=1,\ldots,n} \mathbf{b}_i \geq c\sqrt{n}\right] \leq n \cdot \frac{2}{c^2 n} \leq \frac{2}{c^2}.$$

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**Upshot:** Chebyshev's inequality bounds the maximum load by $O(\sqrt{n})$ with good probability, as compared to $O\left(\frac{\log n}{\log \log n}\right)$ for the direct proof. It is quite loose here.

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Chebyshev's and Markov's inequalities are extremely valuable because they are very general – require few assumptions on the underlying random variable. But by using assumptions, we can often get tighter analysis.