COMPSCI 514: Algorithms for Data Science

Cameron Musco

University of Massachusetts Amherst. Spring 2026.

Lecture 8

Logistics

- Problem Set 2 is due Sunday 3/8 at 11:59pm.
- ~~Problem Set 1 grades will be released later today~~.

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- Problem Set 2 is due Sunday 3/8 at 11:59pm.
- Problem Set 1 grades will be released later today.
- Midterm is **in class** next Thursday, 3/12.
- Study guide is posted on the course webpage (under the midterm row in the schedule tab).
- Past midterms are posted in Canvas. Note that some cover a bit more material than we have seen – e.g., we will not cover the Johnson-Lindenstrauss lemma or low-distortion embeddings before Midterm 1.
- At least some of next Tuesday's lecture will be for midterm review. I will also hold additional review office hours next **Wednesday, 11am-12pm**.
- If you need extended time on the midterm, please reach out by this Friday at the latest to make a plan.

Summary

Last Class:

$X_1 \quad X_2 \quad X_3 \ldots$

- Overview of streaming algorithms.
- The (ϵ, k)-frequent items problem and its applications.
- Count-Min sketch algorithm for frequent items. $- O\left(\frac{k}{\epsilon}\right)$

n items

identify all items that appear $\geq \frac{n}{k}$ times

not identify " $\leq (1-\epsilon)\frac{n}{k}$ times

Summary

Last Class:

- Overview of streaming algorithms.
- The (ϵ, k)-frequent items problem and its applications.
- Count-Min sketch algorithm for frequent items.

This Class:

- Distinct items counting via min-hashing.
- Success probability boosting via the median trick.

Distinct Elements

Distinct Elements (Count-Distinct) Problem: Given a stream $x_1, \ldots, x_n$, estimate the number of distinct elements in the stream.

E.g.,

$$1, 5, 7, 5, 2, 1 \rightarrow 4 \text{ distinct elements}$$

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Applications:

- Distinct IP addresses clicking on an ad or visiting a site.
- Distinct values in a database column (for estimating sizes of joins and group bys).
- Number of distinct search engine queries.
- Counting distinct motifs in large DNA sequences.

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O(n) space
↓
exactly solve
by storing a set

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Google Sawzall, Facebook Presto, Apache Drill, Twitter Algebird

Hashing for Distinct Elements

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Min-Hashing for Distinct Elements (variant of Flajolet-Martin):

- Let $\mathbf{h} : U \to [0, 1]$ be a random hash function (with a real valued output)
- $s := 1$
- For $i = 1, \ldots, n$
  - $s := \min(s, \mathbf{h}(x_i))$
- Return $\tilde{d} = \frac{1}{s} - 1$

s ~ minimum hash value

$h(x_1) = .07381\ldots$ ⇒ s

$h(x_2) = .372\ldots$

$h(x_3) = .73\ldots$

$\vdots$

$h(x_n) = .521\ldots$

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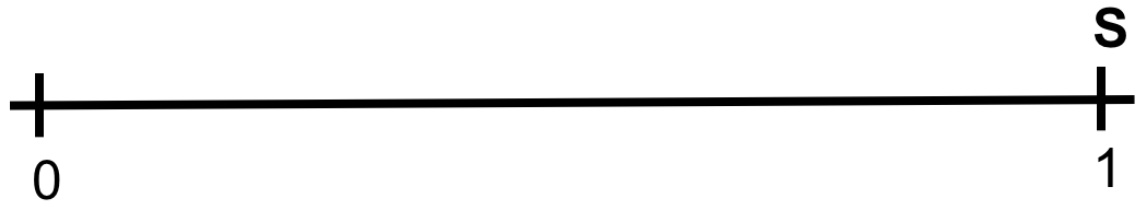

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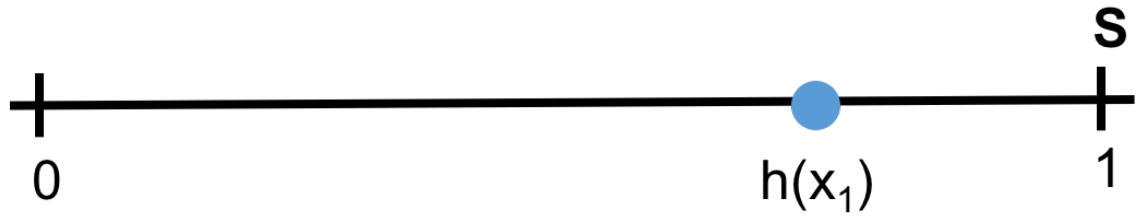

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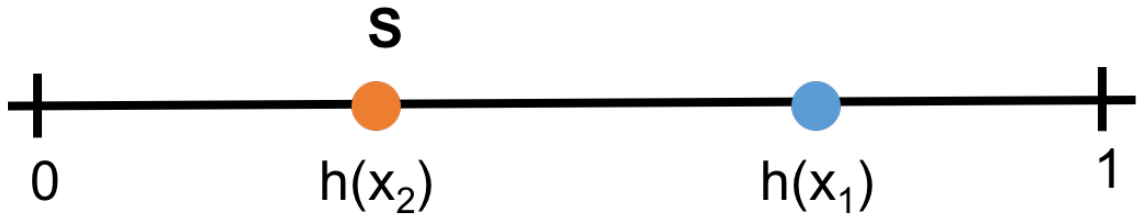

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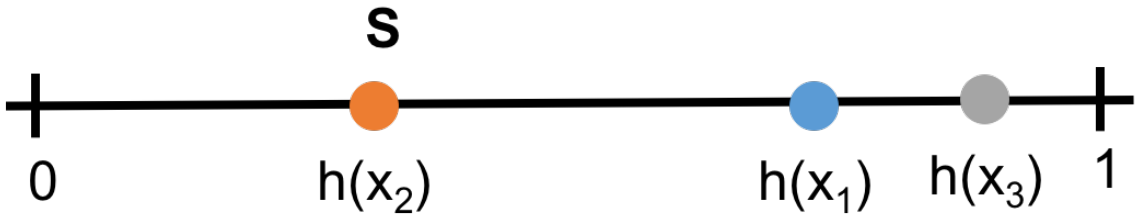

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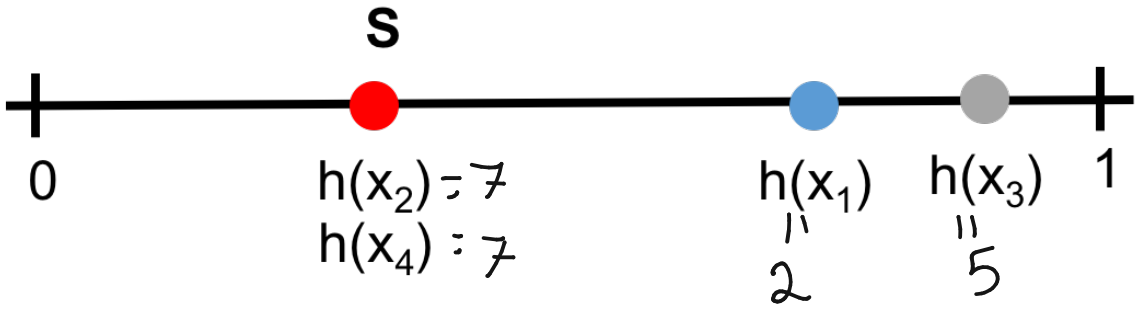

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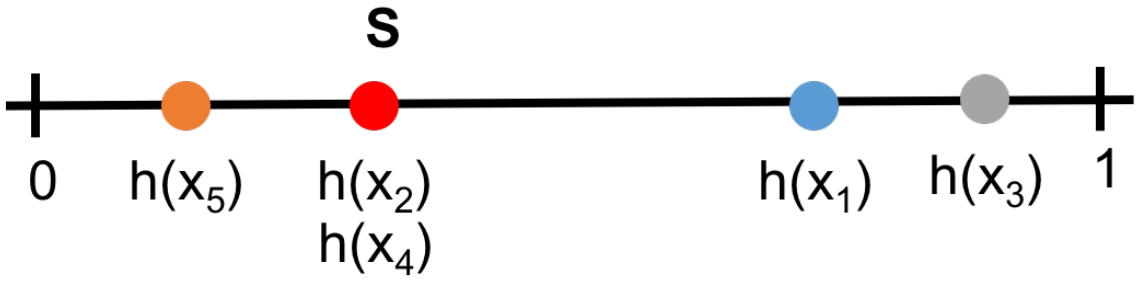

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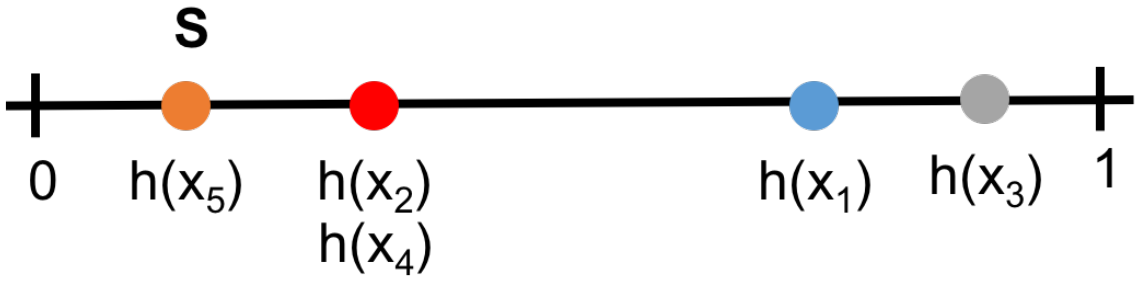

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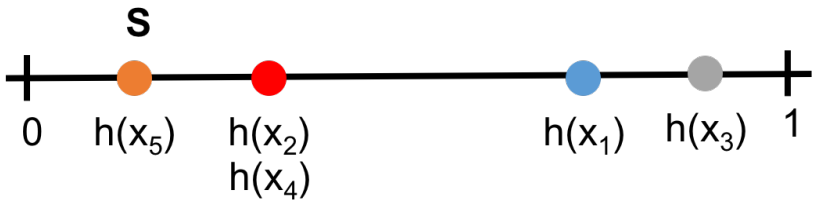

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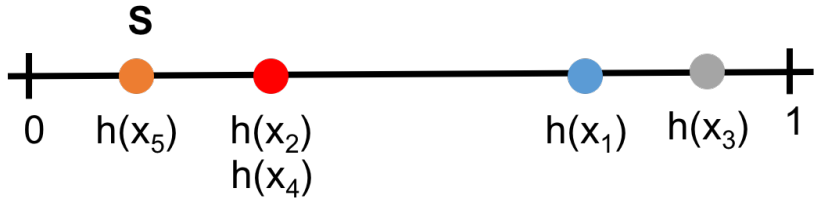

distinct items

- After all items are processed, s is the minimum of d points chosen uniformly at random on $[0, 1]$. Where $d = \#$ distinct elements.

d ↑ s ↓

d ↓ s ↑

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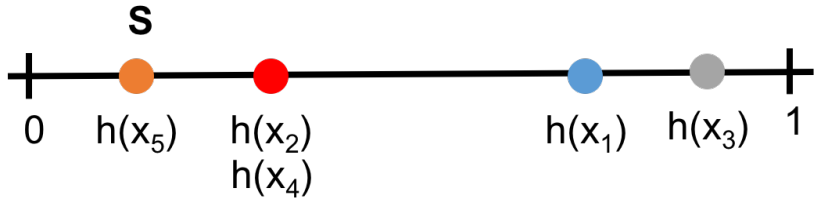

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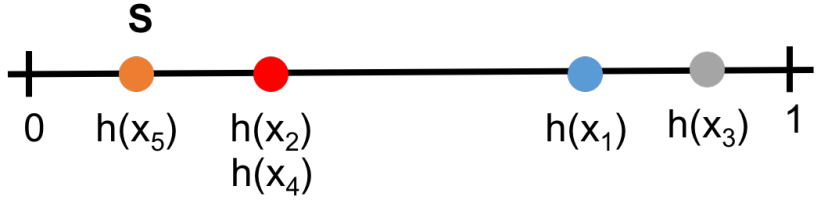

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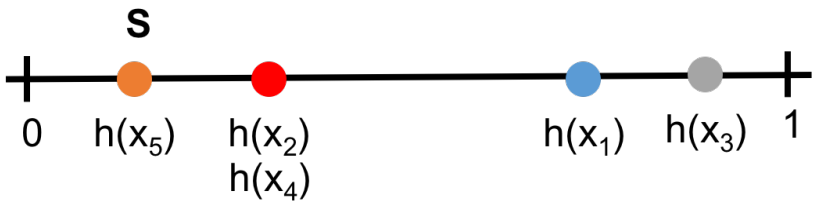
- Intuition: The larger d is, the smaller we expect s to be.
- Same idea as Flajolet-Martin algorithm and HyperLogLog, except they use discrete hash functions.

Performance in Expectation

$\mathbf{s}$ is the minimum of d points chosen uniformly at random on $[0, 1]$.
Where $d = \#$ distinct elements.



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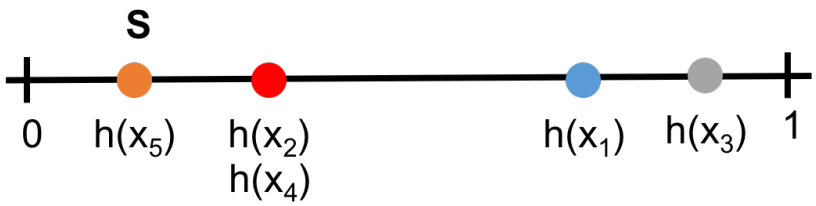
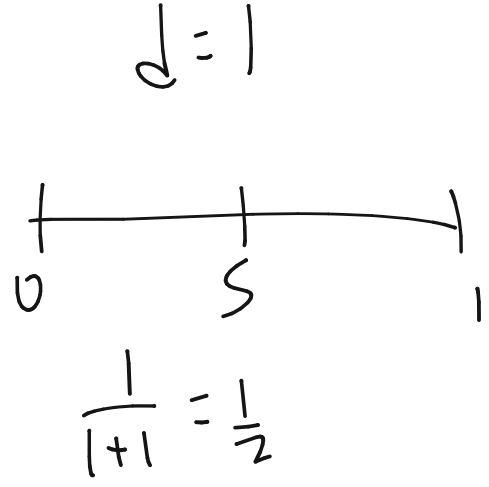
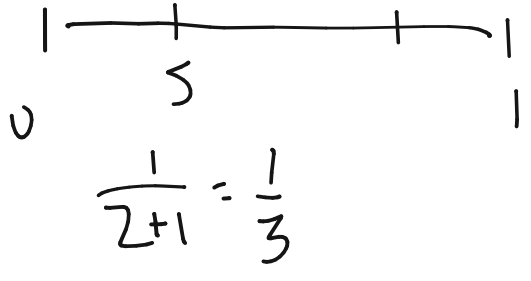


$$\mathbb{E}[\mathbf{s}] = \frac{1}{d+1}$$





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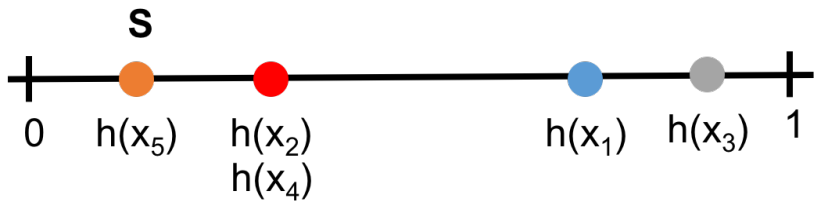
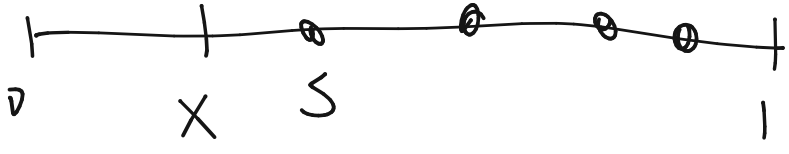


$$\mathbb{E}[\mathbf{s}] = \frac{1}{d+1} \text{ (using } \mathbb{E}(\mathbf{s}) = \int_0^\infty \Pr(\mathbf{s} > x)dx) + \text{calculus)}$$

$$\Pr(s > x) = (1-x)^d$$



$$\int_0^\infty (1-x)^d = \frac{1}{d+1}$$

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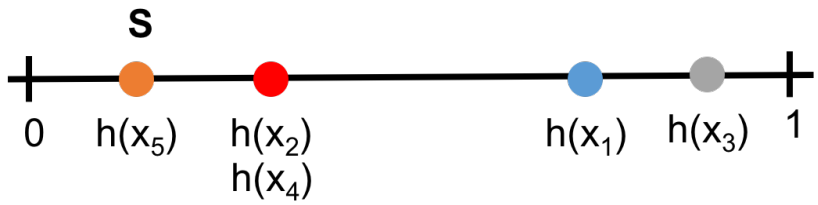

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- So our estimate $\tilde{\mathbf{d}} = \frac{1}{\mathbf{s}} - 1$ is correct if $\mathbf{s}$ exactly equals its expectation.

$$s = \frac{1}{d+1}$$

$$(d+1)s = 1$$

$$d+1 = 1/s \qquad d = 1/s - 1$$

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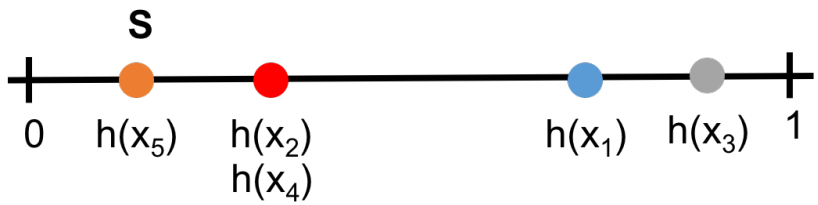

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$$\mathbb{E}\left[\frac{1}{s} - 1\right] = d?$$

$$\text{“} \frac{1}{\mathbb{E}s} - 1 = d$$

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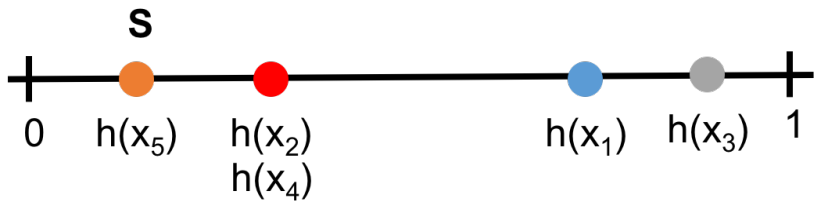

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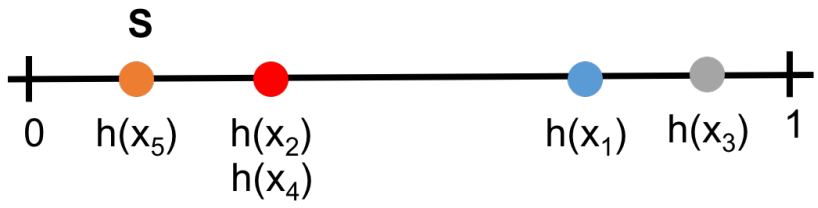

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- **Approximation is robust:** if $|\mathbf{s} - \mathbb{E}[\mathbf{s}]| \leq \epsilon \cdot \mathbb{E}[\mathbf{s}]$ for any $\epsilon \in (0, 1/2)$ and a small constant $c \leq 4$:

$$(1 - c\epsilon)d \leq \widehat{\mathbf{d}} \leq (1 + c\epsilon)d$$

$\epsilon = .01$

$s - \frac{1}{d+1} \leq \epsilon/(d+1)$

$\frac{1}{d+1} - s \leq \epsilon/d+1$

$.96 \cdot d \leq \hat{d} \leq 1.04 \cdot d$

Initial Concentration Bound

So question is how well s concentrates around its mean.

$$\mathbb{E}[\mathsf{s}] = \frac{1}{d+1}$$

.

.

s: minimum of d distinct hashes chosen randomly over $[0,1]$, computed by hashing algorithm. $\widehat{\mathsf{d}} = \frac{1}{\mathsf{s}} - 1$: estimate of # distinct elements d.

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$$\mathbb{E}[\mathsf{s}] = \frac{1}{d+1} \text{ and } \mathsf{Var}[\mathsf{s}] \leq \frac{1}{(d+1)^2} \text{ (\textit{also via calculus}).}$$

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Chebyshev's Inequality:

$$\Pr\left[|\mathsf{s} - \mathbb{E}[\mathsf{s}]| \geq \epsilon \mathbb{E}[\mathsf{s}]\right] \leq \frac{\mathsf{Var}[\mathsf{s}]}{(\epsilon \mathbb{E}[\mathsf{s}])^2} \quad .$$

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Bound is vacuous for any $\epsilon < 1$.

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Bound is vacuous for any $\epsilon < 1$. How can we improve accuracy?

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Improving Performance

Leverage the law of large numbers: improve accuracy via repeated independent trials.

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Hashing for Distinct Elements (Improved):

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  - $\mathsf{s} := \min(\mathsf{s}, \mathsf{h}(x_i))$
- Return $\widehat{\mathsf{d}} = \frac{1}{\mathsf{s}} - 1$

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Leverage the law of large numbers: improve accuracy via repeated independent trials.

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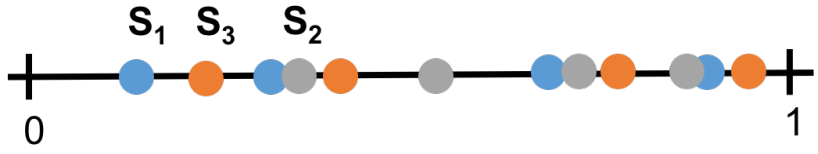

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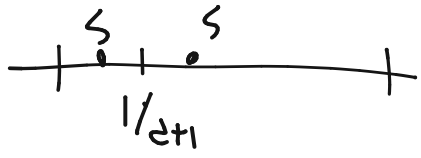

$$\mathbb{E}[\mathsf{s}_j] = \frac{1}{d+1}$$

$$\mathrm{Var}[\mathsf{s}_j] \leq \frac{1}{(d+1)^2}$$

$$\mathbb{E}[s] = \frac{1}{d+1}$$

$$\mathrm{Var}[s] = \mathrm{Var}\left[\frac{1}{k}\sum_{j=1}^{k} s_j\right]$$

$$= \frac{1}{k^2}\sum_{j=1}^{k} \mathrm{Var}(s_j)$$

$$= \frac{1}{k^2} \cdot k \cdot \frac{1}{(d+1)^2} = \frac{1}{k} \cdot \frac{1}{(d+1)^2}$$

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$k \cdot \epsilon^2 = \frac{1}{\delta}$ $\quad k = \frac{1}{\epsilon^2 \delta}$

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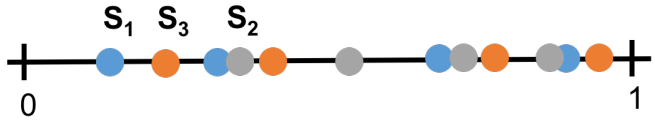

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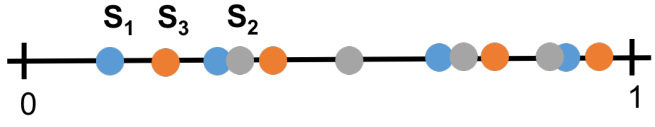

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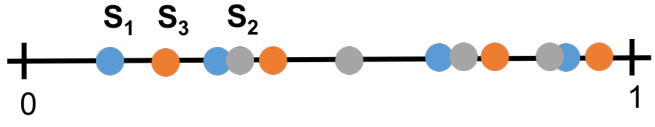

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- $\delta = 5\%$ failure rate gives a factor 20 overhead in space complexity.

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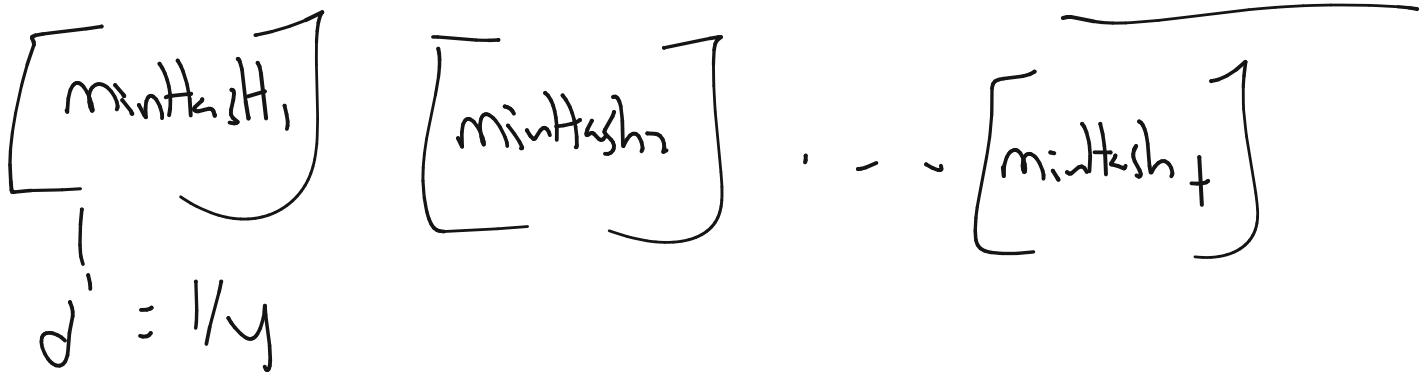

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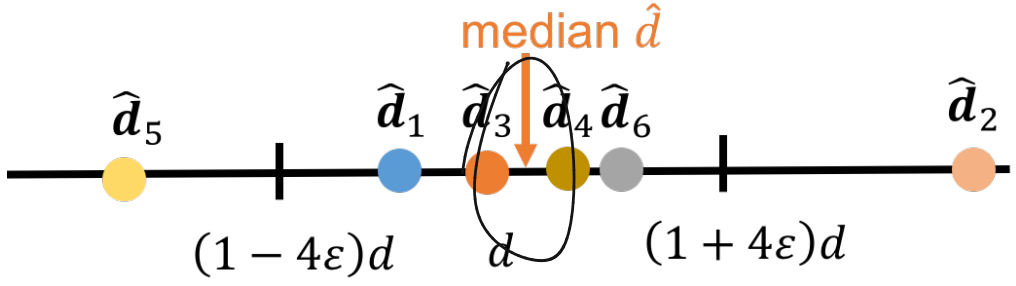

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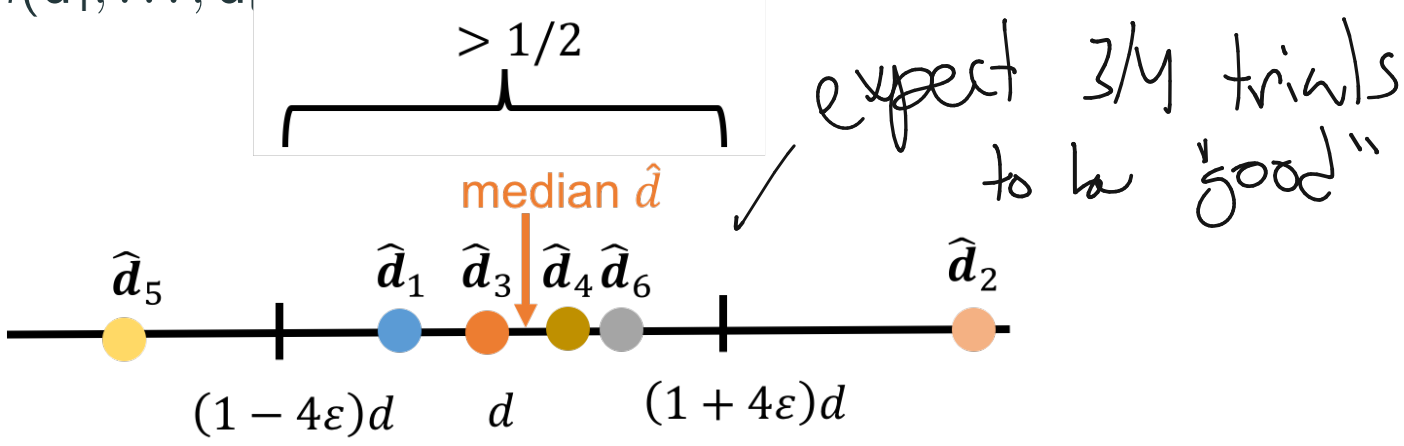

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unrelated to δ'
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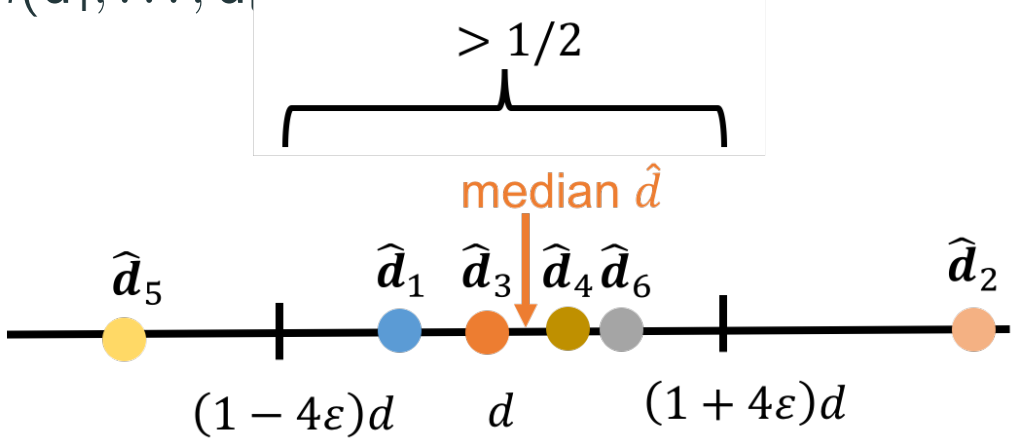

- If $> 1/2$ of trials fall in $[(1-4\epsilon)d, (1+4\epsilon)d]$, then the median will.
- Have $< 1/2$ of trials on both the left and right.

The Median Trick

- $\widehat{\mathbf{d}}_1, \ldots, \widehat{\mathbf{d}}_t$ are the outcomes of the t trials, each falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least $3/4$.
- $\widehat{\mathbf{d}} = median(\widehat{\mathbf{d}}_1, \ldots, \widehat{\mathbf{d}}_t)$.

What is the probability that the median $\widehat{\mathbf{d}}$ falls in $[(1-4\epsilon)d, (1+4\epsilon)d]$?

$\Pr(\widehat{d}$ good estimate$)$

$\widehat{d}$ ← median

$\widehat{d}$ good estimate whenever $\geq \frac{1}{2}$ my trials are good estimates

$\widehat{d}$ bad estimate $\Rightarrow$ $< \frac{1}{2}$ trials are good

$\Pr[X \leq t/2]$ where X # good trials

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$$\Pr\left(\widehat{\mathbf{d}} \notin [(1-4\epsilon)d, (1+4\epsilon)d]\right) \leq \Pr\left(\mathbf{X} < \frac{1}{2} \cdot t\right)$$

$\leq$ small

if $X > \frac{1}{2} t \Rightarrow \hat{d} \in [(1-4\epsilon)d, (1+4\epsilon)d]$

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GOOD BAD GOOD GOOD GOOD

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$$\leq \frac{2}{3} \cdot \frac{3}{4} \cdot t$$

$$\leq \frac{2}{3} \cdot \mathbb{E}[\mathsf{X}]$$

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- $\widehat{\mathsf{d}}_1, \ldots, \widehat{\mathsf{d}}_t$ are the outcomes of the t trials, each falling in $[(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least $3/4$.
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$$c = \frac{\frac{1}{3^2} \cdot \frac{3}{4}}{2 + 1/3}$$

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- Setting $t = O(\log(1/\delta))$ gives failure probability $e^{-\log(1/\delta)} = \delta$.

Median Trick

Upshot: The median of $t = O(\log(1/\delta))$ independent runs of the hashing algorithm for distinct elements returns $\widehat{\mathsf{d}} \in [(1-4\epsilon)d, (1+4\epsilon)d]$ with probability at least $1-\delta$.

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Total Space Complexity: t trials, each using $k = \frac{1}{\epsilon^2 \delta'}$ hash functions, for $\delta' = 1/4$. Space is $\frac{4t}{\epsilon^2} = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$ real numbers (the minimum value of each hash function).

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A note on the median: The median is often used as a robust alternative to the mean, when there are outliers (e.g., heavy tailed distributions, corrupted data).

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The more distinct hashes we see, the higher we expect this maximum to be.

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Note: Careful averaging of estimates from multiple hash functions.

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- Set the maximum # of trailing zeros to the maximum in the two sketches.

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Implementations: Google PowerDrill, Facebook Presto, Twitter Algebird, Amazon Redshift.

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Traditional *COUNT*, *DISTINCT* SQL calls are far too slow, especially when the data is distributed across many servers.

Questions?