COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Spring 2026.
Lecture 6

Logistics

- Problem Set 1 solutions have been posted.
- Problem Set 2 will be posted in the next couple of days and due before the first midterm.

Quiz Poll

Lecture Pacing:

- Way too Fast – 8
- A Bit too Fast – 15
- Just Right – 23
- Too Slow – 1
- Way Too Slow – 0

Last Time

Last Class: $|X_i|$

$M = \max |X_i|$

- Exponential concentration bounds – sums of bounded independent r.v.s. (not necessarily i.i.d)
- Bernstein inequality and the Chernoff bound
- Connection to the central limit theorem.
- Application to random hashing – $O(\log n)$ maximum load per bucket when hashing n items into n buckets.

This Class:

- Bloom filters: random hashing to maintain a large set in small space.

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~~delete(x)~~

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- hash table
- binary tree, linked list, array

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- Allow small probability $\delta > 0$ of false positives. I.e., for any x,

$\delta = .01$

$$\Pr(\underline{query(x)} \underline{= 1} \text{ and } x \notin S) \leq \delta.$$

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Solution: Bloom filters (repeated random hashing). Will use much less space than a hash table.

Bloom Filters

Chose k independent random hash functions $\mathsf{h}_1, \ldots, \mathsf{h}_k$ mapping the universe of elements $U \to [m]$.

- Maintain an array A containing m bits, all initially 0.
- *insert*(x): set all bits $A[\mathsf{h}_1(x)] = \ldots = A[\mathsf{h}_k(x)] := 1$.
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m bit array **A**	0	0	0	0	0	0	0	0	0	0

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Insertions:

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Queries:

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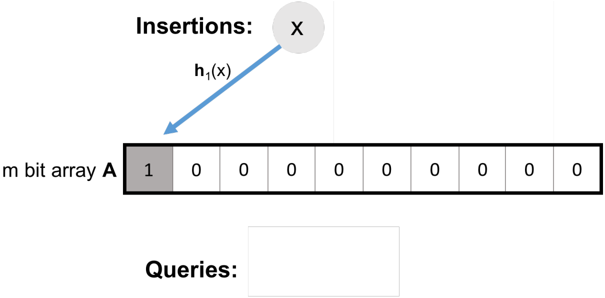

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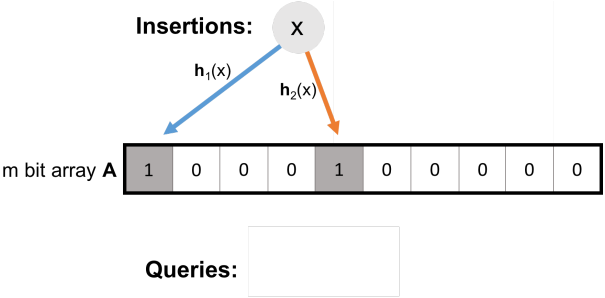

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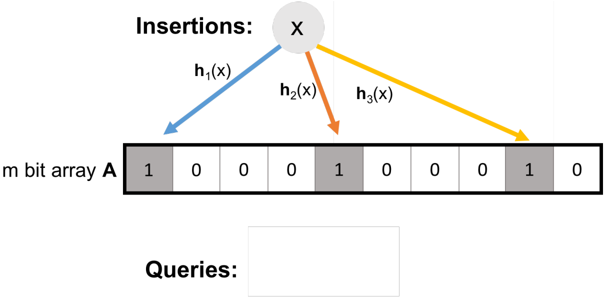
$k=3$



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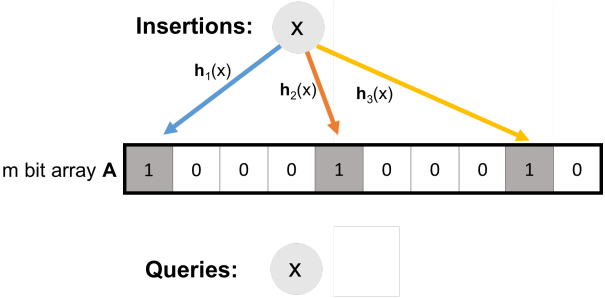

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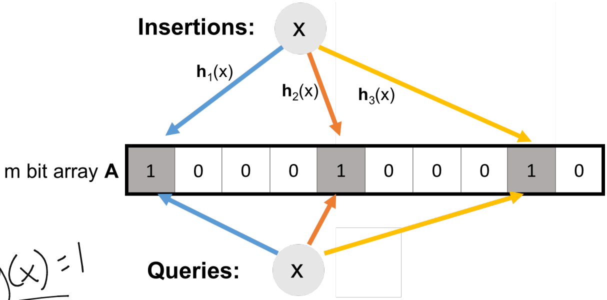

Query(x) = 1

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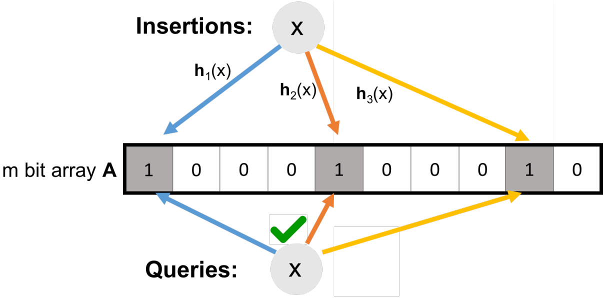

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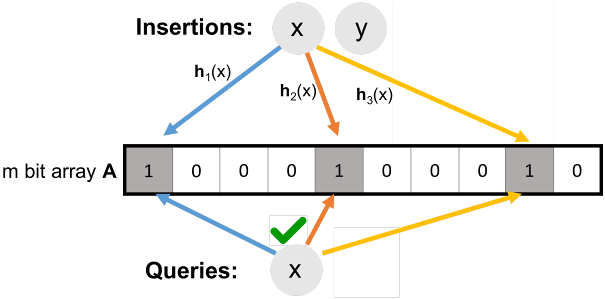

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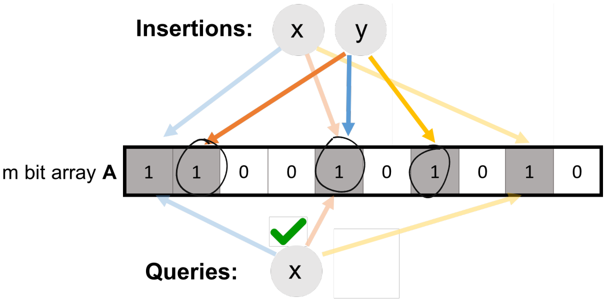

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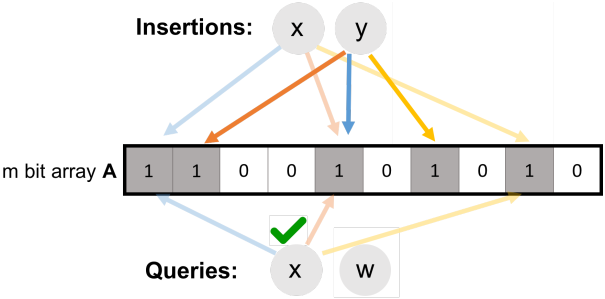

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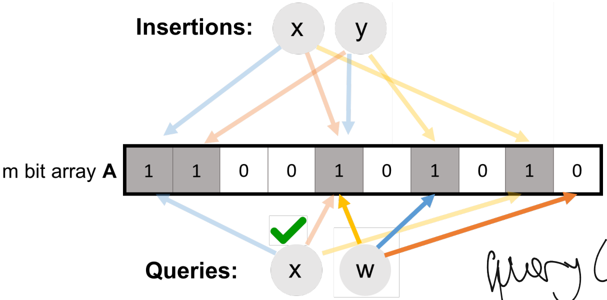

query(w) = 0

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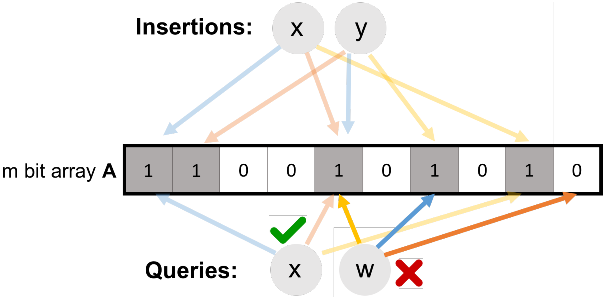

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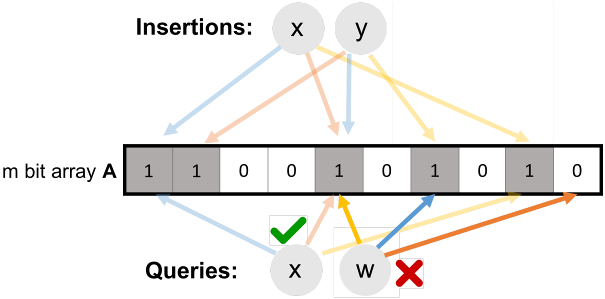
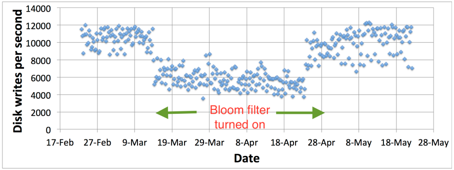


No false negatives. False positives more likely with more insertions.

Akamai (Boston-based company serving 15 – 30% of all web traffic) applies bloom filters to prevent caching of 'one-hit-wonders' – pages only visited once fill over 75% of cache.



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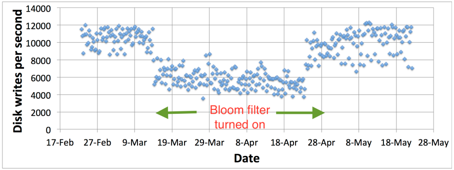

- When url x comes in, if $query(x) = 1$, cache the page at x. If not, run $insert(x)$ so that if it comes in again, it will be cached.

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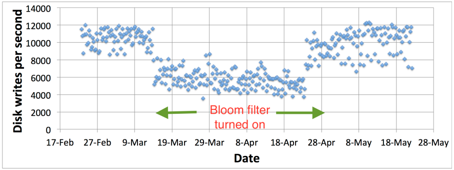

- When url x comes in, if $query(x) = 1$, cache the page at x. If not, run $insert(x)$ so that if it comes in again, it will be cached.
- **False positive:** A new url (possible one-hit-wonder) is cached. If the bloom filter has a false positive rate of $\delta = .05$, the number of cached one-hit-wonders will be reduced by at least 95%.

Applications: Databases

Distributed database systems, including Google Bigtable, Apache HBase, Apache Cassandra, and PostgreSQL use bloom filters to prevent expensive lookups of non-existent data.

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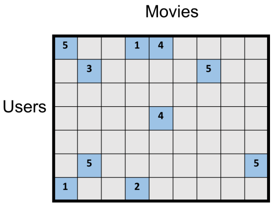

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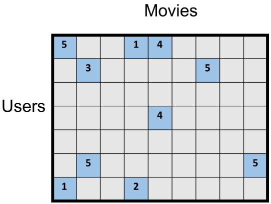

- When a new rating is inserted for $(user_x, movie_y)$, add $(user_x, movie_y)$ to a bloom filter.
- Before reading $(user_x, movie_y)$ (possibly via an out of memory access), check the bloom filter, which is stored in memory.

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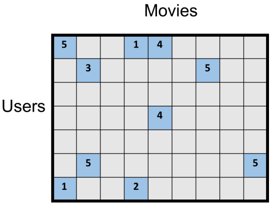

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- Before reading $(user_x, movie_y)$ (possibly via an out of memory access), check the bloom filter, which is stored in memory.
- **False positive:** A read is made to a possibly empty cell. A $\delta = .05$ false positive rate gives a 95% reduction in these empty reads.

More Applications

- Database Joins: Quickly eliminate most keys in one column that don't correspond to keys in another.
- Recommendation systems: Bloom filters are used to prevent showing users the same recommendations twice.
- Spam/Fraud Detection:
  - Bit.ly and Google Chrome use bloom filters to quickly check if a url maps to a flagged site and prevent a user from following it.
  - Can be used to detect repeat clicks on the same ad from a single IP-address, which may be the result of fraud.
- Digital Currency: Some Bitcoin clients use bloom filters to quickly pare down the full transaction log to transactions involving bitcoin addresses that are relevant to them (SPV: simplified payment verification).

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$m \to \infty$

$k \to \infty$

$FPR \to 0$

Analysis

$\left(1-\frac{1}{m}\right)^k \geq 1-\frac{k}{m}$

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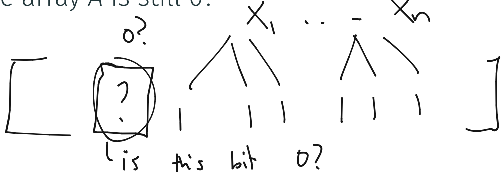

$n = 1$

$k = 1$

$n = 1, k = 1$	$n = 1 \quad k \geq 1$	$n \geq 1, \quad k \geq 1$
$\frac{m-1}{m} = 1 - \frac{1}{m}$	$\left(1-\frac{1}{m}\right)^k$	$\left(1-\frac{1}{m}\right)^{kn}$
	$\cancel{1-\frac{k}{m}}$	

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$$\begin{aligned}\Pr(A[i] = 0) = \Pr\big(&\mathbf{h}_1(x_1) \neq i \cap \ldots \cap \mathbf{h}_k(x_{\cancel{k}1}) \neq i \\ &\underbrace{\cap\, \mathbf{h}_1(x_2) \neq i \ldots \cap \mathbf{h}_k(x_2) \neq i \cap}_{k \cdot n} \ldots\big)\end{aligned}$$

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$$
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&\qquad \left. \cap\, \mathbf{h}_1(x_2) \neq i \ldots \cap \mathbf{h}_k(x_2) \neq i \cap \ldots \right) \\
&= \underbrace{\Pr\left(\mathbf{h}_1(x_1) \neq i\right) \times \ldots \times \Pr\left(\mathbf{h}_k(x_1) \neq i\right) \times \Pr\left(\mathbf{h}_1(x_2) \neq i\right) \ldots}_{k \cdot n \text{ events each occuring with probability } 1 - 1/m}
\end{aligned}
$$

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$x_1 \mid \underline{k=3}$

$$\Pr(A[i]=0) = \Pr\big(\mathbf{h}_1(x_1) \neq i \cap \ldots \cap \mathbf{h}_k(x_k) \neq i$$
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$$= \left(1 - \frac{1}{m}\right)^{kn}$$

$$\left(1 - \frac{m-k}{m}\right)^n$$

$$\left(\frac{k}{m}\right)^n \rightarrow \text{true if } \Pr(\text{inserting 1 item does not } A[i]=1)$$

$$= \cancel{k/m} \quad \left(1 - \frac{1}{m}\right)^k$$

Analysis

How does the false positive rate δ depend on m, k, and the number of items inserted?

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$$\Pr(A[i] = 0) = \left(1 - \frac{1}{m}\right)^{kn}$$

$n \to \infty \quad Pr \to 1$

$k \to \infty \quad Pr \to 0$

$n \to \infty \quad Pr \to 0$

n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

Analysis

How does the false positive rate δ depend on m, k, and the number of items inserted?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0?

$$\left(1-\frac{1}{100}\right)^{100} \approx .37 \approx \frac{1}{e}$$

$$m=2 \qquad \left(1-\frac{1}{2}\right)^2 = \frac{1}{4} \neq \frac{1}{e}$$

$$\Pr(A[i]=0) = \left(1-\frac{1}{m}\right)^{kn} \approx e^{-\frac{kn}{m}}$$

$$\lim_{m \to \infty} \left(1-\frac{1}{m}\right)^m = \frac{1}{e}$$

$$\left(1-\frac{1}{m}\right)^{kn} = \boxed{\left(1-\frac{1}{m}\right)^{m}}^{\frac{kn}{m}} \approx \left(\frac{1}{e}\right)^{\frac{kn}{m}} = e^{-kn/m}$$

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Step 2: What is the probability that querying a new item w gives a false positive?

$$\begin{aligned}\Pr\left(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1\right) \\ = \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1)\end{aligned}$$

$$FPR = \left(1 - e^{-kn/m}\right)^k$$

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n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

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How does the false positive rate δ depend on m, k, and the number of items inserted?

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Correct Analysis Sketch

Step 1: To avoid dependence issues, condition on the event that the A has t zeros in it after n insertions, for some $t \leq m$. For a non-inserted element w, after conditioning on this event we correctly have:

$$\begin{aligned}\Pr(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1) \\ = \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1).\end{aligned}$$

I.e., the events $A[\mathbf{h}_1(w)] = 1$,..., $A[\mathbf{h}_k(w)] = 1$ are independent conditioned on the number of bits set in A. Why?

- Conditioned on this event, for any j, since $\mathbf{h}_j$ is a fully random hash function, $\Pr(A[\mathbf{h}_j(w)] = 1) = 1 - \frac{t}{m}$.
- Thus conditioned on this event, the false positive rate is $\left(1 - \frac{t}{m}\right)^k$.
- It remains to show that $\frac{t}{m} \approx e^{-\frac{kn}{m}}$ with high probability. We already have that $\mathbb{E}[\frac{t}{m}] = \frac{1}{m}\sum_{i=1}^{m} \Pr(A[i] = 0) \approx e^{-\frac{kn}{m}}$.

Correct Analysis Sketch

Need to show that the number of zeros t in A after n insertions is bounded by $O\left(e^{-\frac{kn}{m}}\right)$ with high probability.

Can apply Theorem 2 of:
`http://cglab.ca/~morin/publications/ds/bloom-submitted.pdf`

Analysis

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0?

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$$\text{FPR} = \left(1 - e^{-\frac{kn}{m}}\right)^k$$

n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

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False Positive Rate: with m bits of storage, k hash functions, and n items inserted $\delta \approx \left(1 - e^{\frac{-kn}{m}}\right)^k$.

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$k \uparrow$ filter fills with 1s FPR $\uparrow$

$k \uparrow$ check more positions FRR $\downarrow$

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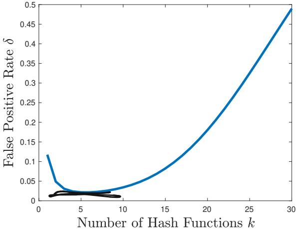

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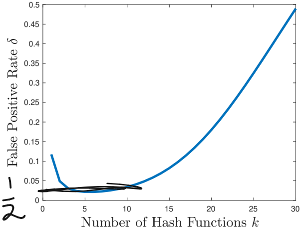
$\delta = \left(1 - e^{-\frac{kn}{m}}\right)^k$

prob. entry = 1

$k = \ln 2 \cdot \frac{m}{n}$

$\text{Prob}(\text{entry} = 1) = \frac{1}{2}$



$k = \ln 2 \cdot \frac{m}{n}$

$FPR^* = \left(1 - e^{-\ln 2 \cdot \frac{m}{n} \cdot \frac{n}{m}}\right)^k$

$= \left(\frac{1}{2}\right)^k$

at optimal Bloom Filter $\approx$ 1/2 full

- Can differentiate to show optimal number of hashes is $k = \ln 2 \cdot \frac{m}{n}$.
- Balances filling up the array vs. having enough hashes so that even when the array is pretty full, a new item is unlikely to yield a false positive.

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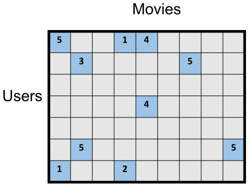

10^9 (user, movie) pairs

- Say we have 100 million users, each who have rated 10 movies.
- $n = 10^9 = n$ (user,movie) pairs with non-empty ratings.

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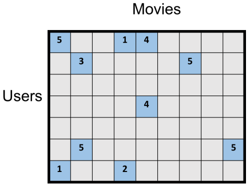

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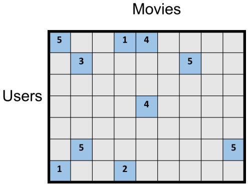

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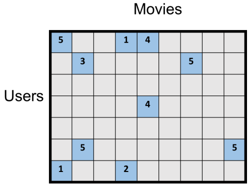

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- Set $k = \ln 2 \cdot \frac{m}{n} = 5.54 \approx 6$.
- False positive rate is $\approx \left(1 - e^{-k \cdot \frac{n}{m}}\right)^k \approx \frac{1}{2^k} \approx \frac{1}{2^{5.54}} = .021$.

Bloom Filter Note

An observation about Bloom filter space complexity:

$$\text{False Positive Rate: } \delta \approx \left(1 - e^{-\frac{kn}{m}}\right)^k.$$

For an m-bit bloom filter holding n items, optimal number of hash functions k is: $k = \ln 2 \cdot \frac{m}{n}$.

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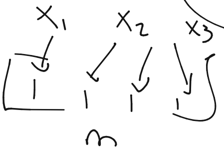
Think Pair Share: If we want a false positive rate $< \frac{1}{2}$ how big does m need to be in comparison to n?

$$m = O(\log n), \; m = O(\sqrt{n}), m = O(n), \; m = O(n^2)?$$



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I.e., storing n items in a bloom filter requires $O(n)$ space. So what's the point? Truly $O(n)$ bits, rather than $O(n \cdot \text{item size})$.

Questions on Bloom Filters?