COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Spring 2020.
Lecture 24 (Final Lecture!)

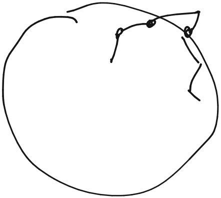
LOGISTICS

- Problem Set 4 is due Sunday 5/3 at 8pm.
- Exam is at **2pm on May 6th**. Open note, similar to midterm.
- Exam review guide and practice problems have been posted under the schedule tab on the course page.
- I will hold usual office hours today and exam review office hours this Thursday and next Tuesday during the regular class time 11:30am-12:45pm
- Regular SRTI's are suspended this semester. But I am holding an optional SRTI for this class and would really appreciate your feedback.
- `http://owl.umass.edu/partners/courseEvalSurvey/uma/`.

SUMMARY

Last Class:

- Analysis of gradient descent for optimizing convex functions.
- (The same) analysis of projected gradient descent for optimizing under (convex) constraints. $\theta \in \mathbb{R}^d$ $\theta \in S \subseteq \mathbb{R}^d$
- Convex sets and projection functions.



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- Convex sets and projection functions.

This Class:

- Online learning, regret, and online gradient descent.
- Application to analysis of stochastic gradient descent (if time).
- Course summary/wrap-up

In reality many learning problems are online.

- Websites optimize ads or recommendations to show users, given continuous feedback from these users.
- Spam filters are incrementally updated and adapt as they see more examples of spam over time.
- Face recognition systems, other classification systems, learn from mistakes over time.

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Want to minimize some global loss $L(\vec{\theta}, \mathbf{X}) = \sum_{i=1}^{n} \ell(\vec{\theta}, \vec{x}_i)$, when data points are presented in an online fashion $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ (like in streaming algorithms)

ONLINE GRADIENT DESCENT

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$\nabla L(\theta, X) = \sum_{i=1}^{n} \nabla \ell(\theta, x_i)$

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$x_1 \ldots x_n \; x_{n+1}$

Stochastic gradient descent is a special case: when data points are considered a random order for computational reasons.

Computation

Online Optimization: In place of a single function f, we see a different objective function at each step: $f_i = \ell(\theta, x_i)$

$$f_1, \ldots, f_t : \mathbb{R}^d \to \mathbb{R}$$

ONLINE OPTIMIZATION FORMAL SETUP

ML setup: $\ell(\theta, x_i) = f_i(\theta)$

Online Optimization: In place of a single function f, we see a different objective function at each step:

$$f_1, f_2, \ldots, f_t : \mathbb{R}^d \rightarrow \mathbb{R}$$

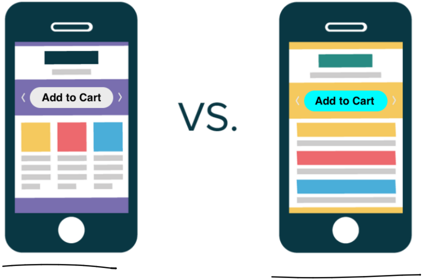
$f_1(\theta) = \ell(\theta, x_1)$ $\quad \ell(\theta, x_2)$

$\theta_1 \rightarrow f_1(\theta_1)$, $\theta_2 \rightarrow f_2(\theta_2)$

- At each step, first pick (play) a parameter vector $\vec{\theta}^{(i)}$.
- Then are told f_i and incur cost $f_i(\vec{\theta}^{(i)})$.
- **Goal:** Minimize total cost $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)})$.

No assumptions on how $f_1, \ldots, f_t$ are related to each other!

UI design via online optimization.



- Parameter vector $\vec{\theta}^{(i)}$: some encoding of the layout at step i.
- Functions $f_1, \ldots, f_t$: $f_i(\vec{\theta}^{(i)}) = 1$ if user does not click 'add to cart' and $f_i(\vec{\theta}^{(i)}) = 0$ if they do click.
- Want to maximize number of purchases. I.e., minimize $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)})$

ONLINE OPTIMIZATION EXAMPLE

Home pricing tools.

$\vec{x} = [\#baths, \#beds, \#floors \ldots]$

linear model

$\langle \vec{x}, \vec{\theta} \rangle$

→ $275,000

- Parameter vector $\vec{\theta}^{(i)}$: coefficients of linear model at step i.
- Functions $f_1, \ldots, f_t$: $f_i(\vec{\theta}^{(i)}) = (\langle \vec{x}_i, \vec{\theta}^{(i)} \rangle - price_i)^2$ revealed when $home_i$ is listed or sold.
- Want to minimize total squared error $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)})$ (same as classic least squares regression).

offline

In normal optimization, we seek $\hat{\theta}$ satisfying:

$$f(\hat{\theta}) \leq \min_{\vec{\theta}} f(\vec{\theta}) + \epsilon. \equiv f(\theta^*) + \epsilon$$

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$$\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)}) \leq \min_{\vec{\theta}} \sum_{i=1}^{t} f_i(\vec{\theta}) + \epsilon = \sum_{i=1}^{t} f_i(\vec{\theta}^{off}) + \epsilon$$

ϵ is called the regret.

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$\varepsilon = 0$

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$\varepsilon \geq 0$

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- This error metric is a bit 'unfair'. Why?
- Comparing online solution to best fixed solution in hindsight. ϵ can be negative!

$f_1 \ldots f_t$

$f_1(\theta) = |\theta - 1000|$, $f_2(\theta) = |\theta + 1000|$ . . . .

What if for $i = 1, \ldots, t$, $f_i(\theta) = |x - 1000|$ or $f_i(\theta) = |x + 1000|$ in an alternating pattern?

How small can the regret ϵ be? $\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)}) \leq \sum_{i=1}^{t} f_i(\vec{\theta}^{off}) + \epsilon$.

$\sum_{i=1}^{t} f_i(\vec{\theta}^{(i)}) = 0$, $\sum_{i=1}^{t} f_i(\vec{\theta}^{off}) = 1000t$

$\min_\theta \sum_{i=1}^{t} f_i(\theta) = 1000t$

$\theta^1 = 1000$ $\theta^2 = -1000$ $\theta^3 = 1000$ . . .

ε can be negative

Choose θ^i before seeing f_i

θ^i calculated using past $f_1(\theta_1)$ $f_2(\theta_2)$. ... $f_{i-1}(\theta^{i-1})$

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What if for $i = 1, \ldots, t$, $f_i(\theta) = |x - 1000|$ or $f_i(\theta) = |x + 1000|$ in no particular pattern? How can any online learning algorithm hope to achieve small regret?

Assume that:

- $f_1, \ldots, f_t$ are all convex.
- Each f_i is G-Lipschitz (i.e., $\|\vec{\nabla} f_i(\vec{\theta})\|_2 \leq G$ for all $\vec{\theta}$.)
- $\|\vec{\theta}^{(1)} - \vec{\theta}^{off}\|_2 \leq R$ where $\theta^{(1)}$ is the first vector chosen.

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Online Gradient Descent

- Set step size $\eta = \frac{R}{G\sqrt{t}}$. Pick some initial θ^1
- For $i = 1, \ldots, t$
  - Play $\vec{\theta}^{(i)}$ and incur cost $f_i(\vec{\theta}^{(i)})$.
  - $\vec{\theta}^{(i+1)} = \vec{\theta}^{(i)} - \eta \cdot \vec{\nabla} f_i(\vec{\theta}^{(i)})$

Theorem – OGD on Convex Lipschitz Functions: For convex G-Lipschitz $f_1, \ldots, f_t$, OGD initialized with starting point $\theta^{(1)}$ within radius R of θ^{off}, using step size $\eta = \frac{R}{G\sqrt{t}}$, has regret bounded by:

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Step 1.1: For all i, $\nabla f_i(\theta^{(i)})(\theta^{(i)} - \theta^{off}) \leq \frac{\|\theta^{(i)} - \theta^{off}\|_2^2 - \|\theta^{(i+1)} - \theta^{off}\|_2^2}{2\eta} + \frac{\eta G^2}{2}$.

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Convexity $\implies$ **Step 1:** For all i,

$$f_i(\theta^{(i)}) - f_i(\theta^{off}) \leq \frac{\|\theta^{(i)} - \theta^{off}\|_2^2 - \|\theta^{(i+1)} - \theta^{off}\|_2^2}{2\eta} + \frac{\eta G^2}{2}.$$

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$$\left[\sum_{i=1}^{t} f_i(\theta^{(i)}) - \sum_{i=1}^{t} f_i(\theta^{off})\right] \leq \sum_{i=1}^{t} \frac{\|\theta^{(i)} - \theta^{off}\|_2^2 - \|\theta^{(i+1)} - \theta^{off}\|_2^2}{2\eta} + \frac{t \cdot \eta G^2}{2}.$$

telescoping

$$= \frac{\|\theta^{1} - \theta^{off}\|_2^2 - \|\theta^{t+1} - \theta^{off}\|_2^2}{2\eta} + \frac{t\eta G^2}{2} \leq \frac{R^2}{2\eta} + \frac{t\eta G^2}{2}$$

Stochastic gradient descent is an efficient offline optimization method, seeking $\hat{\theta}$ with

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- The most popular optimization method in modern machine learning.
- Easily analyzed as a special case of online gradient descent!

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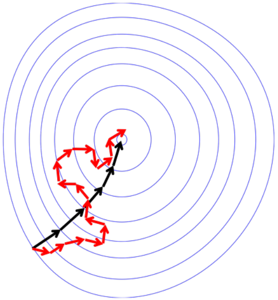
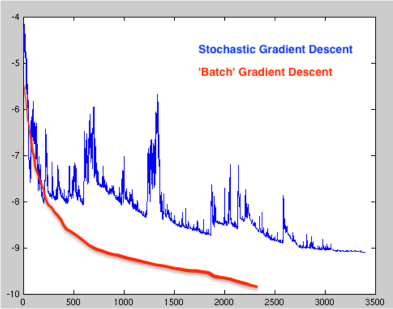
Stochastic Gradient Descent

- Set step size $\eta = \frac{R}{G\sqrt{t}}$.
- For $i = 1, \ldots, t$
  - Pick random $j_i \in 1, \ldots, n$.
  - $\vec{\theta}^{(i+1)} = \vec{\theta}^{(i)} - \eta \cdot \vec{\nabla} f_{j_i}(\vec{\theta}^{(i)})$
- Return $\hat{\theta} = \frac{1}{t} \sum_{i=1}^{t} \vec{\theta}^{(i)}$.

STOCHASTIC GRADIENT DESCENT



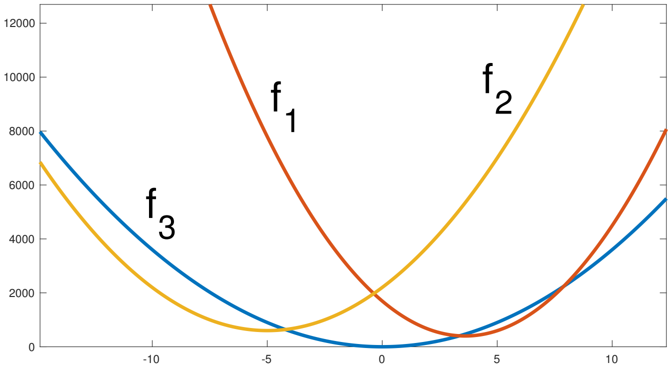


$$\vec{\theta}^{(i+1)} = \vec{\theta}^{(i)} - \eta \cdot \vec{\nabla} f_{j_i}(\vec{\theta}^{(i)}) \text{ vs. } \vec{\theta}^{(i+1)} = \vec{\theta}^{(i)} - \eta \cdot \vec{\nabla} f(\vec{\theta}^{(i)})$$

Note that: $\mathbb{E}[\vec{\nabla} f_{j_i}(\vec{\theta}^{(i)})] = \frac{1}{n} \vec{\nabla} f(\vec{\theta}^{(i)})$.

Analysis extends to any algorithm that takes the gradient step in expectation (batch GD, randomly quantized, measurement noise, differentially private, etc.)

What does $f_1(\theta) + f_2(\theta) + f_3(\theta)$ look like?



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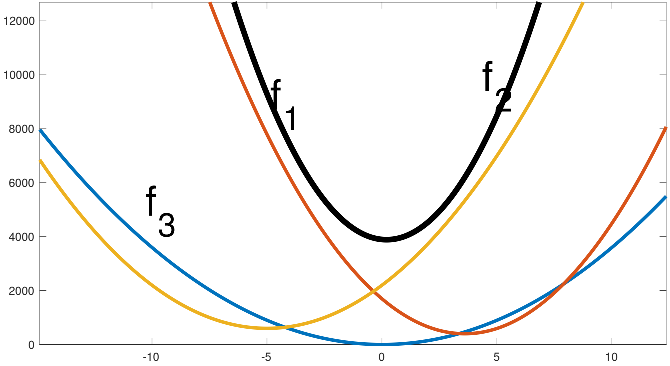

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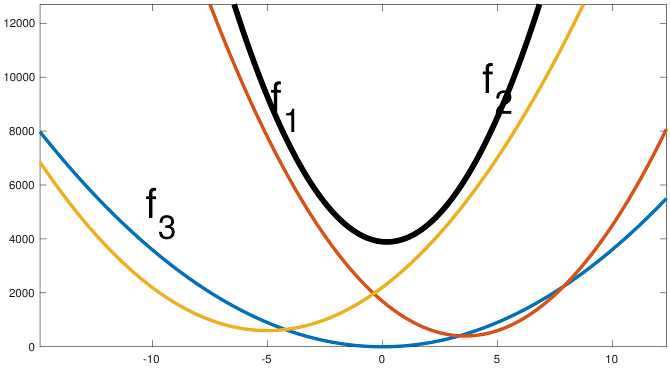

A sum of convex functions is always convex (good exercise).

Theorem – SGD on Convex Lipschitz Functions: SGD run with $t \geq \frac{R^2G^2}{\epsilon^2}$ iterations, $\eta = \frac{R}{G\sqrt{t}}$, and starting point within radius R of θ^*, outputs $\hat{\theta}$ satisfying: $\mathbb{E}[f(\hat{\theta})] \leq f(\theta^*) + \epsilon$.

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Step 2: $\mathbb{E}[f(\hat{\theta}) - f(\theta^*)] \leq \frac{n}{t} \cdot \mathbb{E}\left[\sum_{i=1}^{t}[f_{j_i}(\theta^{(i)}) - f_{j_i}(\theta^*)]\right]$.

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Step 3: $\mathbb{E}[f(\hat{\theta}) - f(\theta^*)] \leq \frac{n}{t} \cdot \mathbb{E}\left[\sum_{i=1}^{t}[f_{j_i}(\theta^{(i)}) - f_{j_i}(\theta^{off})]\right].$

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Step 4: $\mathbb{E}[f(\hat{\theta}) - f(\theta^*)] \leq \frac{n}{t} \cdot \underbrace{R \cdot \frac{G}{n} \cdot \sqrt{t}}_{\text{OGD bound}} = \frac{RG}{\sqrt{t}}.$

Stochastic gradient descent generally makes more iterations than gradient descent.

Each iteration is much cheaper (by a factor of n).

$$\vec{\nabla} \sum_{j=1}^{n} f_j(\vec{\theta}) \text{ vs. } \vec{\nabla} f_j(\vec{\theta})$$

When $f(\vec{\theta}) = \sum_{j=1}^{n} f_j(\vec{\theta})$ and $\|\vec{\nabla} f_j(\vec{\theta})\|_2 \leq \frac{G}{n}$:

Theorem – SGD: After $t \geq \frac{R^2 G^2}{\epsilon^2}$ iterations outputs $\hat{\theta}$ satisfying:

$$\mathbb{E}[f(\hat{\theta})] \leq f(\theta^*) + \epsilon.$$

When $\|\vec{\nabla} f(\vec{\theta})\|_2 \leq \bar{G}$:

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$\|\vec{\nabla} f(\vec{\theta})\|_2 = \|\vec{\nabla} f_1(\vec{\theta}) + \ldots + \vec{\nabla} f_n(\vec{\theta})\|_2 \leq \sum_{j=1}^{n} \|\vec{\nabla} f_j(\vec{\theta})\|_2 \leq n \cdot \frac{G}{n} \leq G.$

SGD VS. GD

When $f(\vec{\theta}) = \sum_{j=1}^{n} f_j(\vec{\theta})$ and $\|\vec{\nabla} f_j(\vec{\theta})\|_2 \leq \frac{G}{n}$:

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$$\mathbb{E}[f(\hat{\theta})] \leq f(\theta^*) + \epsilon.$$

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Theorem – GD: After $t \geq \frac{R^2 \bar{G}^2}{\epsilon^2}$ iterations outputs $\hat{\theta}$ satisfying:

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$\|\vec{\nabla} f(\vec{\theta})\|_2 = \|\vec{\nabla} f_1(\vec{\theta}) + \ldots + \vec{\nabla} f_n(\vec{\theta})\|_2 \leq \sum_{j=1}^{n} \|\vec{\nabla} f_j(\vec{\theta})\|_2 \leq n \cdot \frac{G}{n} \leq G.$

When would this bound be tight?

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- Just the tip of the iceberg on randomized streaming/sketching/hashing algorithms.
- In the process covered probability/statistics tools that are very useful beyond algorithm design: concentration inequalities, higher moment bounds, law of large numbers, central limit theorem, linearity of expectation and variance, union bound, median as a robust estimator.

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- Spectral graph theory – nonlinear dimension reduction and spectral clustering for community detection.
- In the process covered linear algebraic tools that are very broadly useful in ML and data science: eigendecomposition, singular value decomposition, projection, norm transformations.

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- Online optimization and online gradient descent.
- Lots that we didn't cover: stochastic gradient descent, accelerated methods, adaptive methods, second order methods (quasi-Newton methods), practical considerations. Gave mathematical tools to understand these methods.

Thanks for a great semester!