COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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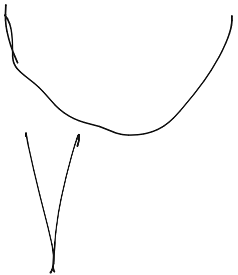
University of Massachusetts Amherst. Spring 2020.

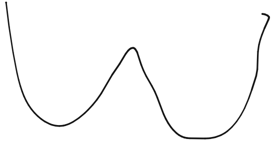
Lecture 23

SUMMARY

Last Class:

- Multivariable calculus review and gradient computation.
- Introduction to gradient descent. Motivation as a greedy algorithm.
- Conditions under which we will analyze gradient descent: convexity and Lipschitzness.





Last Class:

- Multivariable calculus review and gradient computation.
- Introduction to gradient descent. Motivation as a greedy algorithm.
- Conditions under which we will analyze gradient descent: convexity and Lipschitzness.

This Class:

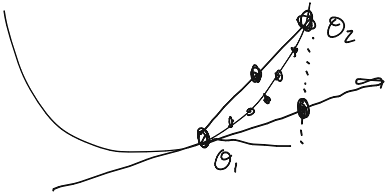
- Analysis of gradient descent for Lipschitz, convex functions.
- Simple extension to projected gradient descent for constrained optimization. $f(\theta)$ s.t $\theta \in S$

Definition – Convex Function: A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if and only if, for any $\vec{\theta}_1, \vec{\theta}_2 \in \mathbb{R}^d$ and $\lambda \in [0, 1]$:

$$(1-\lambda) \cdot f(\vec{\theta}_1) + \lambda \cdot f(\vec{\theta}_2) \geq f\left((1-\lambda) \cdot \vec{\theta}_1 + \lambda \cdot \vec{\theta}_2\right)$$

Corollary – Convex Function: A function $f : \mathbb{R}^d \to \mathbb{R}$ is convex if and only if, for any $\vec{\theta}_1, \vec{\theta}_2 \in \mathbb{R}^d$ and $\lambda \in [0, 1]$:

$$f(\vec{\theta}_2) - f(\vec{\theta}_1) \geq \vec{\nabla} f(\vec{\theta}_1)^T \left(\vec{\theta}_2 - \vec{\theta}_1\right)$$



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Definition – Lipschitz Function: A function $f : \mathbb{R}^d \to \mathbb{R}$ is G-Lipschitz if $\|\vec{\nabla} f(\vec{\theta})\|_2 \leq G$ for all $\vec{\theta}$.

Assume that: $\vec{\theta}_* = \arg\min_{\vec{\theta}} f(\vec{\theta})$

- f is convex.
- f is G-Lipschitz.
- $\|\vec{\theta}_1 - \vec{\theta}_*\|_2 \leq R$ where $\vec{\theta}_1$ is the initialization point.

Gradient Descent

- Choose some initialization $\vec{\theta}_1$ and set $\underline{\eta = \frac{R}{G\sqrt{t}}}$.
- For $i = 1, \ldots, t-1$
  - $\underline{\vec{\theta}_{i+1}} = \underline{\vec{\theta}_i} - \underline{\eta \vec{\nabla} f(\vec{\theta}_i)}$
- Return $\hat{\theta} = \arg\min_{\underline{\vec{\theta}_1, \ldots, \vec{\theta}_t}} f(\vec{\theta}_i)$.

Theorem – GD on Convex Lipschitz Functions: For convex G-Lipschitz function f, GD run with $t \geq \frac{R^2 G^2}{\epsilon^2}$ iterations, $\eta = \frac{R}{G\sqrt{t}}$, and starting point within radius R of $\vec{\theta}_*$, outputs $\hat{\theta}$ satisfying:

$$f(\hat{\theta}) \leq f(\vec{\theta}_*) + \epsilon.$$

GD ANALYSIS PROOF

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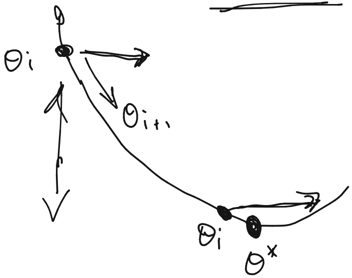
Step 1: For all i, $f(\vec{\theta}_i) - f(\vec{\theta}_*) \leq \frac{\|\vec{\theta}_i - \vec{\theta}_*\|_2^2 - \|\vec{\theta}_{i+1} - \vec{\theta}_*\|_2^2}{2\eta} + \frac{\eta G^2}{2}$. Visually:



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$\|a-b\|_2^2 = \|a\|_2^2 + \|b\|_2^2 - 2a^Tb$

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$$\|\theta_{i+1} - \theta_*\|_2^2 = \|\theta_i - \eta \nabla f(\theta_i) - \theta_*\|_2^2$$

$$= \|\theta_i - \theta_*\|_2^2 + \underbrace{\|\eta \nabla f(\theta_i)\|_2^2}_{} - 2\eta \nabla f(\theta_i)^T(\theta_i - \theta_*)$$

$$\|\eta \nabla f(\theta_i)\|_2^2 \leq \eta^2 G^2$$

$$\|\theta_{i+1} - \theta_*\|_2^2 \leq \|\theta_i - \theta_*\|_2^2 + \eta^2 G^2 - 2\eta \nabla f(\theta_i)^T(\theta_i - \theta_*)$$

$$\nabla f(\theta_i)^T(\theta_i - \theta_*) \leq \frac{\|\theta_i - \theta_*\|_2^2 - \|\theta_{i+1} - \theta_*\|_2^2}{2\eta} + \frac{\eta^2 G^2}{2\eta}$$

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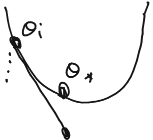
RHS

convexity

$f(\theta_i) - f(\theta_*) \leq \nabla f(\theta_i)^T(\theta_i - \theta_*) \leq \text{RHS}$



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Step 2: $\frac{1}{t}\sum_{i=1}^{t} f(\vec{\theta}_i) - f(\vec{\theta}_*) \leq \underline{\frac{R^2}{2\eta \cdot t}} + \underline{\frac{\eta G^2}{2}} \leq \varepsilon$

want: $\min_{i=1,\dots,t} f(\theta_i) - f(\theta_*) \leq \varepsilon$

By Step 1:

$$\frac{1}{t}\sum f(\theta_i) - f(\theta_*) \leq \frac{\eta G^2}{2} + \frac{1}{t}\sum\left[\frac{\|\theta_i - \theta_*\|^2 - \|\theta_{i+1} - \theta_*\|_2^2}{2\eta}\right]$$

$$\|\theta_1 - \theta^*\|^2 - \cancel{\|\theta_2 - \theta^*\|^2} + \cancel{\|\theta_2 - \theta_*\|} - \cancel{\|\theta_3 - \theta_*\|^2} + \cancel{\dots} \cancel{\|\theta_t - \theta_*\|} - \|\theta_{t+1} - \theta_*\|$$

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distance you haven't covered

overshoot

Step 2: $\frac{1}{t}\sum_{i=1}^{t} f(\vec{\theta}_i) - f(\vec{\theta}_*) \leq \frac{R^2}{2\eta\cdot t} + \frac{\eta G^2}{2}.$

$$\frac{\eta G^2}{2} + \frac{1}{t}\sum_{i=1}^{t} \frac{\|\theta_i - \theta_*\|_2^2 - \|\theta_{i+1} - \theta_*\|_2^2}{2\eta} = \frac{\eta G^2}{2} + \frac{\|\theta_1 - \theta_*\|_2^2 - \|\theta_{t+1} - \theta_*\|_2^2}{2\eta \cdot t}$$

$$\leq \frac{\eta G^2}{2} + \frac{R^2 - \|\theta_{t+1} - \theta_*\|_2^2}{2\eta t}$$

$$\leq \frac{\eta G^2}{2} + \frac{R^2}{2\eta t}$$

$$\eta = \frac{R}{G\sqrt{t}}$$

$$\leq \frac{RG}{2\sqrt{t}} + \frac{R^2}{2\frac{R}{G\sqrt{t}}\cdot t} = \frac{RG}{2\sqrt{t}} + \frac{RG}{2\sqrt{t}} = \frac{RG}{\sqrt{t}} \leq \epsilon$$

Often want to perform convex optimization with convex constraints.

$\theta \in \mathbb{R}^d$

$$\vec{\theta}^* = \underset{\vec{\theta} \in \mathcal{S}}{\arg\min} f(\vec{\theta}),$$

$\|\theta\|_2^2 < 1$

$\theta \in$ subspace V.

where $\mathcal{S}$ is a convex set.

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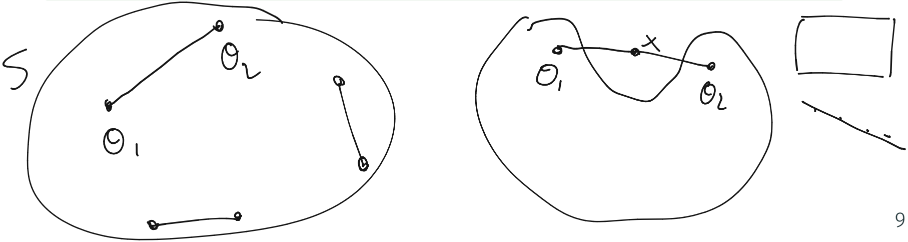
Definition – Convex Set: A set $\mathcal{S} \subseteq \mathbb{R}^d$ is convex if and only if, for any $\vec{\theta}_1, \vec{\theta}_2 \in \mathcal{S}$ and $\lambda \in [0, 1]$:

$$(1 - \lambda)\vec{\theta}_1 + \lambda \cdot \vec{\theta}_2 \in \mathcal{S}$$



CONSTRAINED CONVEX OPTIMIZATION

Often want to perform convex optimization with convex constraints.

$$\vec{\theta}^* = \arg\min_{\vec{\theta} \in \mathcal{S}} f(\vec{\theta}),$$

① $\|a+b\| \leq \|a\| + \|b\|$

② $\|a-b\| \leq \|a\| + \|b\|$

$\|a-(-b)\| \leq \|a\| + \|-b\|$

where $\mathcal{S}$ is a convex set.

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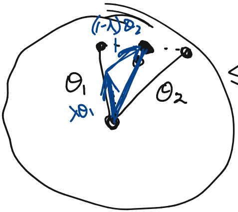
$$(1-\lambda)\vec{\theta}_1 + \lambda \cdot \vec{\theta}_2 \in \mathcal{S}$$

E.g. $\mathcal{S} = \{\vec{\theta} \in \mathbb{R}^d : \|\vec{\theta}\|_2 \leq 1\}$.

prove this is convex

$\|\theta_1\| \leq 1$, $\|\theta_2\| \leq 1$

$\|\lambda\theta_1 + (1-\lambda)\theta_2\| \leq 1$



$\|\lambda\theta_1 + (1-\lambda)\theta_2\|_2$

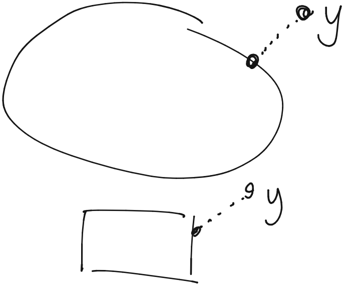
$\leq \lambda\|\theta_1\| + (1-\lambda)\|\theta_2\|_2$

$\leq \lambda \cdot 1 + (1-\lambda) \cdot 1$

$= 1$

For any convex set let $P_{\mathcal{S}}(\cdot)$ denote the projection function onto $\mathcal{S}$.

- $P_{\mathcal{S}}(\vec{y}) = \arg\min_{\vec{\theta} \in \mathcal{S}} \|\vec{\theta} - \vec{y}\|_2$.



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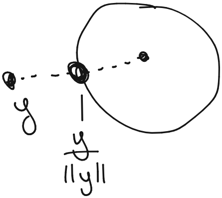
- For $\mathcal{S} = \{\vec{\theta} \in \mathbb{R}^d : \|\vec{\theta}\|_2 \leq 1\}$ what is $P_{\mathcal{S}}(\vec{y})$?

$$P_S(y) = \frac{y}{\|y\|_2}$$



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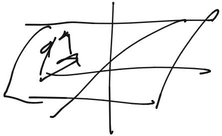
- For $\mathcal{S} = \{\vec{\theta} \in \mathbb{R}^d : \|\vec{\theta}\|_2 \leq 1\}$ what is $P_{\mathcal{S}}(\vec{y})$?
- For $\mathcal{S}$ being a k dimensional subspace of $\mathbb{R}^d$, what is $P_{\mathcal{S}}(\vec{y})$?



$S = \{\theta : \theta = Vc\}$

$P_s(y) = VV^T y$

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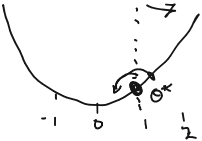


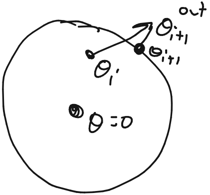
$\min f(\theta)$

$\theta \in [1,2]$

$\|\theta\|_2^2$ s.t. $\|\theta\|_2 \leq 1$

Projected Gradient Descent

- Choose some initialization $\vec{\theta}_1$ and set $\eta = \frac{R}{G\sqrt{t}}$.
- For $i = 1, \ldots, t-1$
  - $\vec{\theta}_{i+1}^{(out)} = \vec{\theta}_i - \eta \cdot \vec{\nabla} f(\vec{\theta}_i)$
  - $\vec{\theta}_{i+1} = P_{\mathcal{S}}(\vec{\theta}_{i+1}^{(out)})$.
- Return $\hat{\theta} = \arg\min_{\vec{\theta}_i} f(\vec{\theta}_i)$.



Projected gradient descent can be analyzed identically to gradient descent!

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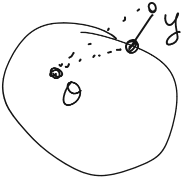
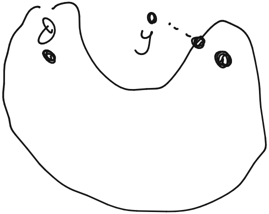
Theorem – Projection to a convex set: For any convex set $\mathcal{S} \subseteq \mathbb{R}^d$, $\vec{y} \in \mathbb{R}^d$, and $\vec{\theta} \in \mathcal{S}$,

$$\|P_{\mathcal{S}}(\vec{y}) - \vec{\theta}\|_2 \leq \|\vec{y} - \vec{\theta}\|_2.$$





Theorem – Projected GD: For convex G-Lipschitz function f, and convex set $\mathcal{S}$, Projected GD run with $t \geq \frac{R^2 G^2}{\epsilon^2}$ iterations, $\eta = \frac{R}{G\sqrt{t}}$, and starting point within radius R of $\vec{\theta}_*$, outputs $\hat{\theta}$ satisfying:

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Recall: $\vec{\theta}_{i+1}^{(out)} = \underline{\vec{\theta}_i} - \eta \cdot \vec{\nabla} f(\vec{\theta}_i)$ and $\underline{\vec{\theta}_{i+1} = P_{\mathcal{S}}(}\vec{\theta}_{i+1}^{(out)})$.

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$\|\theta_{i+1} - \theta_*\| \leq \|\theta_{i+1}^{out} - \theta^*\|$

[, , ,]

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Step 2: $\frac{1}{t}\sum_{i=1}^{t} f(\vec{\theta}_i) - f(\vec{\theta}_*) \leq \frac{R^2}{2\eta \cdot t} + \frac{\eta G^2}{2}$ $\implies$ Theorem.

- Grad Descent: convex / Lipschitz
- constrained opt. over convex set via projected GD
- Project GD follow directly GD.