

COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2026.

Lecture 5

- Problem Set 1 is due next Friday at 11:59pm.
- Try to start early and attend office hours if you feel stuck on any of the questions.
- Online Quiz 3 will be posted immediately after class today and due Monday at 9am.

Last Time

Last Class:

$$\Pr(|X - \mathbb{E}X| > t) \leq \frac{\text{Var}(X)}{t^2}$$

- Chebyshev's inequality and the law of large numbers.
- The union bound.
- Application to hashing for load balancing.
- Start on exploring stronger concentration bounds.

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This Time:

- Higher moment bounds $\rightarrow$ exponential concentration bounds and the **central limit theorem**.

Flipping Coins

We flip $n = 100$ independent coins, each are heads with probability $1/2$ and tails with probability $1/2$. Let H be the number of heads.

$$\mathbb{E}[H] = \frac{n}{2} = 50 \text{ and } \text{Var}[H] = \frac{n}{4} = 25$$

Markov's:	Chebyshev's:	In Reality:
$\Pr(H \geq 60) \leq .833$	$\Pr(H \geq 60) \leq .25$	$\Pr(H \geq 60) = 0.0284$
$\Pr(H \geq 70) \leq .714$	$\Pr(H \geq 70) \leq .0625$	$\Pr(H \geq 70) = .000039$
$\Pr(H \geq 80) \leq .625$	$\Pr(H \geq 80) \leq .0278$	$\Pr(H \geq 80) < 10^{-9}$

- H has a simple Binomial distribution, so can compute these probabilities exactly.
- Markov and Chebyshev's inequalities are extremely **general** and so can be loose – we would like to incorporate more information about the underlying distribution to get tighter bounds.

Higher Moment Bounds

- Markov's: $\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. First Moment.

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- What if we just apply Markov's inequality to even higher moments?

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- What if we just apply Markov's inequality to even higher moments?
- E.g., a fourth moment bound: Consider any random variable X :

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right)$$

4th moment of X

Higher Moment Bounds

- Markov's: $\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. **First Moment.**
- Chebyshev's: $\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr((X - \mathbb{E}[X])^2 \geq t^2) \leq \frac{\text{Var}[X]}{t^2}$.
Second Moment.
- What if we just apply Markov's inequality to even higher moments?
- E.g., a fourth moment bound: Consider any random variable X :

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr((X - \mathbb{E}[X])^4 \geq t^4) \leq \frac{\mathbb{E}[(X - \mathbb{E}[X])^4]}{t^4}.$$

need to compute this

A Fourth Moment Bound

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right) \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{t^4}.$$

Application to Coin Flips: Recall: $n = 100$ independent fair coins, H is the number of heads.

- Bound the fourth moment:

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Application to Coin Flips: Recall: $n = 100$ independent fair coins, H is the number of heads.

- Bound the fourth moment:

$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right]$$

where $H_i = 1$ if coin flip i is heads and 0 otherwise.

A Fourth Moment Bound

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Application to Coin Flips: Recall: $n = 100$ independent fair coins, H is the number of heads.

$$(H_1 + H_2 + H_3 + H_4)^4 = H_1^4 + H_1^3 H_2 \cdot 3 + H_1^2 H_2 H_3 \cdot ?$$

- Bound the fourth moment:

$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right] = \sum_{i,j,k,\ell} c_{ijkl} \underbrace{\mathbb{E}[H_i H_j H_k H_\ell]}_{\substack{\text{if flip } i,j,k,\ell \\ \text{are heads, 0} \\ \text{otherwise}}}$$

$$i \neq j \neq k \neq \ell \quad \mathbb{E}[H_i H_j H_k H_\ell] = \frac{1}{16}$$

A Fourth Moment Bound

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where $H_i = 1$ if coin flip i is heads and 0 otherwise. Then apply some messy calculations...

A Fourth Moment Bound

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right) \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{t^4} \leq \frac{1862.5}{t^4} \leq \frac{25}{t^2}$$

Application to Coin Flips: Recall: $n = 100$ independent fair coins, H is the number of heads.

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$H_1 H_2 H_3^2 = H_1 H_2 H_3 H_3$

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where $H_i = 1$ if coin flip i is heads and 0 otherwise. Then apply some messy calculations...

- Apply Fourth Moment Bound: $\Pr(|H - \mathbb{E}[H]| \geq t) \leq \frac{1862.5}{t^4}.$

Tighter Bounds

Chebyshev's:

$$\Pr(H \geq 60) \leq .25$$

$$\Pr(H \geq 70) \leq .0625$$

$$\Pr(H \geq 80) \leq .04$$

$$\frac{25}{t^2}$$

4th Moment:

$$\Pr(H \geq 60) \leq .186$$

$$\Pr(H \geq 70) \leq .0116$$

$$\Pr(H \geq 80) \leq .0023$$

$$\frac{186.5}{t^4}$$

In Reality:

$$\Pr(H \geq 60) = 0.0284$$

$$\Pr(H \geq 70) = .000039$$

$$\Pr(H \geq 80) < 10^{-9}$$

H: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$.

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Can we just keep applying Markov's inequality to higher and higher moments and getting tighter bounds?

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$$\leq \frac{\mathbb{E}[|X - \mathbb{E}X|^k]}{t^k}$$

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$$a \geq b \Leftrightarrow f(a) \geq f(b)$$

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• Why monotonic?

$$\Pr(|X - \mathbb{E}[X]| > t) = \Pr(\underbrace{f(|X - \mathbb{E}[X]|)}_{\text{(positive)}} \underbrace{> f(t)}_{\mathbb{R}}) \leq \frac{\mathbb{E}f(|X - \mathbb{E}[X]|)}{f(t)}$$

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Exponential Concentration Bounds

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Moment Generating Function: Consider for any $t > 0$:

$f(x)$

$$M_t(X) = e^{t \cdot (X - \mathbb{E}[X])}$$

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- $M_t(X)$ is monotonic for any $t > 0$.

↓ positive

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- Choosing t appropriately lets one prove a number of very powerful **exponential concentration bounds** (exponential tail bounds).

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- Chernoff bound, Bernstein inequalities, Hoeffding's inequality, Azuma's inequality, Berry-Esseen theorem, etc.

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- Choosing t appropriately lets one prove a number of very powerful **exponential concentration bounds** (exponential tail bounds).
- Chernoff bound, Bernstein inequalities, Hoeffding's inequality, Azuma's inequality, Berry-Esseen theorem, etc.
- We will not cover the proofs in this class – just the statements.

Bernstein Inequality

$$|X_i| < m \quad \forall i$$

$$e^{-\infty} = 0 \quad e^{-0} = 1$$

Bernstein Inequality: Consider **independent** random variables $X_1, \dots, X_n$ all falling in $[-M, M]$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \text{Var}[X_i]$. For any $t \geq 0$:

$$\Pr \left(\left| \sum_{i=1}^n X_i - \mu \right| \geq t \right) \leq 2 \exp \left(- \frac{t^2}{2\sigma^2 + \frac{4}{3}Mt} \right).$$

$$2 \exp(-R)$$

$t \uparrow$

gets more negative (better bound)

$\sigma^2 \uparrow$

gets less negative (worse bound)

$m \uparrow$

get less negative (worse bound)

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Assume that $M = 1$ and plug in $\underline{t = s \cdot \sigma}$ for $\underline{s \leq \sigma}$.

Bernstein Inequality

Bernstein Inequality: Consider **independent** random variables $X_1, \dots, X_n$ all falling in $[-1, 1]$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \text{Var}[X_i]$. For any $s \geq 0$:

$$\Pr \left(\left| \sum_{i=1}^n X_i - \mu \right| \geq s\sigma \right) \leq 2 \exp \left(-\frac{s^2}{4} \right).$$

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Compare to Chebyshev's: $\Pr (|\sum_{i=1}^n X_i - \mu| \geq s\sigma) \leq \frac{1}{s^2}$.

$$\frac{6^2}{(56)^2} = \frac{1}{5^2}$$

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- An exponentially stronger dependence on s !

Comparison to Chebyshev's

Consider again bounding the number of heads H in $n = 100$ independent coin flips.

Chebyshev's:

$$\Pr(H \geq 60) \leq .25$$

$$\Pr(H \geq 70) \leq .0625$$

$$\Pr(H \geq 80) \leq .04$$

Bernstein:

$$\Pr(H \geq 60) \leq .21$$

$$\Pr(H \geq 70) \leq .005$$

$$\Pr(H \geq 80) \leq 4^{-5}$$

In Reality:

$$\Pr(H \geq 60) = 0.0284$$

$$\Pr(H \geq 70) = .000039$$

$$\Pr(H \geq 80) < 10^{-9}$$

H : total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$.

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Getting much closer to the true probability.

H : total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$.

The Chernoff Bound

A useful variation of the Bernstein inequality for binary (indicator) random variables is:

Chernoff Bound (simplified version): Consider independent random variables $X_1, \dots, X_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$. For any $\delta \geq 0$

$$\Pr \left(\left| \sum_{i=1}^n X_i - \mu \right| \geq \delta \mu \right) \leq 2 \exp \left(-\frac{\delta^2 \mu}{2 + \delta} \right).$$

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- As δ gets larger and larger, the bound falls off exponentially fast.

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$$\Pr \left(\left| \sum_{i=1}^n X_i - \mu \right| \geq \delta \mu \right) \leq 2 \exp \left(-\frac{\delta^2 \mu}{2 + \delta} \right).$$

- As δ gets larger and larger, the bound falls off exponentially fast.
- How does this set up differ from a Binomial distribution?

↳ X_i are not necessarily identically distributed

Interpretation as a Central Limit Theorem

Bernstein Inequality (Simplified): Consider independent random variables $X_1, \dots, X_n$ falling in $[-1,1]$. Let $\mu = \mathbb{E}[\sum X_i]$, $\sigma^2 = \text{Var}[\sum X_i]$, and $s \leq \sigma$. Then:

$$\Pr \left(\left| \sum_{i=1}^n X_i - \mu \right| \geq s\sigma \right) \leq 2 \exp \left(-\frac{s^2}{4} \right).$$

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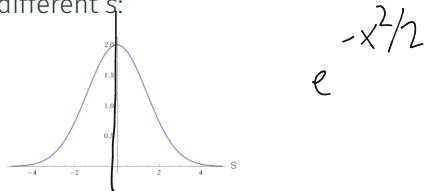
Can plot this bound for different s :

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Can plot this bound for different s :

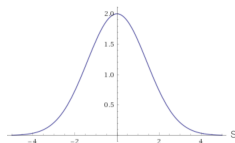


Interpretation as a Central Limit Theorem

Bernstein Inequality (Simplified): Consider independent random variables $X_1, \dots, X_n$ falling in $[-1,1]$. Let $\mu = \mathbb{E}[\sum X_i]$, $\sigma^2 = \text{Var}[\sum X_i]$, and $s \leq \sigma$. Then:

$$\Pr \left(\left| \sum_{i=1}^n X_i - \mu \right| \geq s\sigma \right) \leq 2 \exp \left(-\frac{s^2}{4} \right).$$

Can plot this bound for different s :



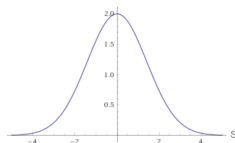
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$-s^2/4$
 $2e$

Essentially the same bound that Bernstein's inequality gives!

Gaussian Tails

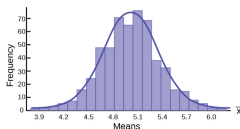
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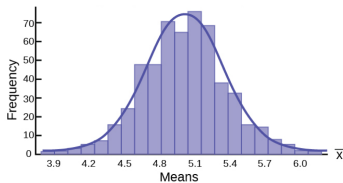
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Central Limit Theorem Interpretation: Bernstein's inequality gives a quantitative version of the CLT. The distribution of the sum of *bounded* independent random variables can be upper bounded with a Gaussian (normal) distribution.



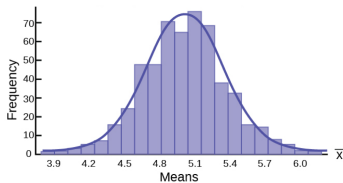
Central Limit Theorem

Stronger Central Limit Theorem: The distribution of the sum of n *bounded* independent random variables converges to a Gaussian (normal) distribution as n goes to infinity.



Central Limit Theorem

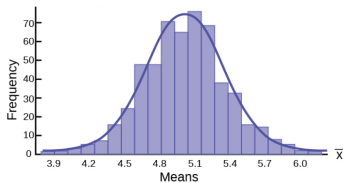
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- Why is the Gaussian distribution is so important in statistics, science, ML, etc.?

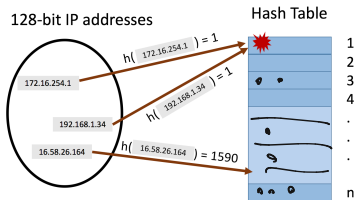
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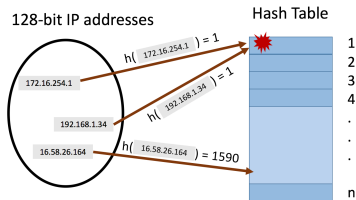
- Why is the Gaussian distribution is so important in statistics, science, ML, etc.?
- Many random variables can be approximated as the sum of a large number of small and roughly independent random effects. Thus, their distribution looks Gaussian by CLT.

Return to Random Hashing



We hash m values $x_1, \dots, x_m$ using a random hash function into a table with $n = m$ entries.

Return to Random Hashing

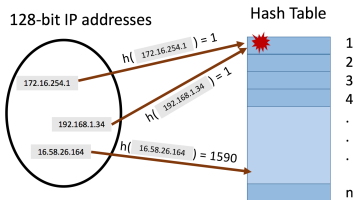


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- I.e., for all $j \in [m]$ and $i \in [m]$, $\Pr(h(x_j) = i) = \frac{1}{m}$ and hash values are chosen independently.

Return to Random Hashing

$$\begin{aligned}
 \Pr(S_i \geq 1) &= 1 - \Pr(S_i = 0) \\
 &= 1 - \left(1 - \frac{1}{m}\right)^m \\
 m \rightarrow \infty \quad \left(1 - \frac{1}{m}\right)^m &\rightarrow \frac{1}{e} \\
 1 - \frac{1}{e} &\approx .7 \\
 S_i &= \text{Hicks in slot } i
 \end{aligned}$$



$$\mathbb{H}[S_i] = 1$$

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What will be the maximum number of items hashed into the same location, with probability $\geq .99$?

$$O(m)$$

$$O(\sqrt{m})$$

$$O(\log m)$$

$$O(1)$$

Maximum Load in Randomized Hashing

Let S_i be the number of items hashed into position i and $S_{i,j}$ be 1 if x_j is hashed into bucket i ($h(x_j) = i$) and 0 otherwise.

m : total number of items hashed and size of hash table. $x_1, \dots, x_m$: the items.
 h : random hash function mapping $x_1, \dots, x_m \rightarrow [m]$.

Maximum Load in Randomized Hashing

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$$S_i = \sum_{j=1}^m S_{i,j}$$

$$\mathbb{E}[S_i] = \sum_{j=1}^m \mathbb{E}[S_{i,j}] = m \cdot \frac{1}{m} = 1$$

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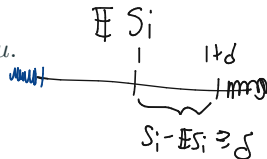
$\Pr[|S_i - \mathbb{E}[S_i]|]$ independent indicator variables
(Chernoff, Bernstein)

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By the Chernoff Bound: for any $\delta \geq 0$,

($\delta = O(\log m)$)

$$\Pr(S_i \geq 1 + \delta) \leq \Pr\left(\left|\sum_{j=1}^n S_{i,j} - 1\right| \geq \delta \cdot \mu\right) \leq 2 \exp\left(-\frac{\delta^2}{2 + \delta}\right)$$

(probability of "many items in bucket i") (Chernoff)

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$$S_i \geq 1 + \delta \Rightarrow S_i - \mathbb{E}[S_i] \geq \delta \Rightarrow |S_i - \mathbb{E}[S_i]| \geq \delta$$

$$\Pr(S_i \geq 1 + \delta) \leq \Pr(S_i - \mathbb{E}[S_i] \geq \delta) \leq \Pr(\dots)$$

Maximum Load in Randomized Hashing

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$$\Pr(S_i \geq 20 \log m + 1) \leq 2 \exp\left(-\frac{(20 \log m)^2}{2 + 20 \log m}\right) \leq 2 \exp\left(-\frac{200 (\log m)^2}{22 (\log m)}\right)$$

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↓
basically 0

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- Using Chebyshev's inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability (good exercise).
- The Chebyshev bound holds even with a pairwise independent hash function. The stronger Chernoff-based bound can be shown to hold with a *k-wise independent hash function* for $k = O(\log m)$.

Questions on Exponential Concentration Bounds?

This concludes the probability foundations part of the course –
on to algorithms.