COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2026.

Lecture 3

Logistics

- Remember to sign up for Gradescope. Needed to see your in-class quiz grades. (Quiz 1 is graded.)
- Second Canvas quiz will be posted after class. Second in-class quiz will be next Monday.

Content Overview

Last Class:

- Application of linearity of expectation to analyzing expected number of collisions in a hash table.
- Markov's inequality: the most fundamental concentration bound.

$$\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}.$$

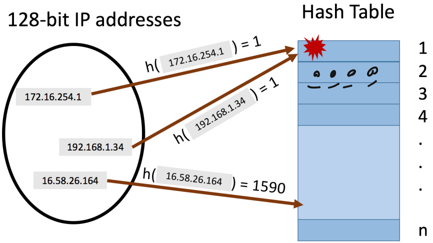
Content Overview

Today:

- Application of Markov's inequality to collision free hashing.
- 2-level hashing for optimal lookup time and storage. – linearity expectation
- 2-universal and pairwise independent hash functions.
- Possibly start on application of random hashing to distributed load balancing.
- Through this application learn about Chebyshev's inequality, which strengthens Markov's inequality. ✓
- Connection to the law of large numbers.

Hash Tables

We store m items from a large universe in a hash table with n positions.



- Want to show that when $\mathbf{h} : U \to [n]$ is a fully random hash function, query time is $O(1)$ with good probability.
- Equivalently: want to show that there are few collisions between hashed items.

Collision Free Hashing

Application: Collision Free Hashing

Via linearity of expectation, we shows that the expected number of pairwise collisions is:

$\binom{m}{2} \cdot \frac{1}{n}$

$$\mathbb{E}[C] = \frac{m(m-1)}{2n}.$$

m: total number of stored items, n: hash table size, C: total pairwise collisions.

Application: Collision Free Hashing

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- Say we have a lot of space. In particular, let $\underline{n = 4m^2}$. Then: $\mathbb{E}[C] = \frac{m(\cancel{m}-1)}{8m^{\cancel{2}}} \leq \frac{1}{8}$.

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$\sum C_{ij} = C$

$\mathbb{E}[(C - \mathbb{E}C)^2]$

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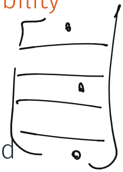

Apply Markov's Inequality: $\Pr[\mathrm{C} \geq 1] \leq \frac{\mathbb{E}[\mathrm{C}]}{1} = \frac{1}{8}$.

So with probability at least 7/8 we have no collisions at all and worst-case $O(1)$ query time.

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Very fast...but we are using $O(m^2)$ space to store m items...

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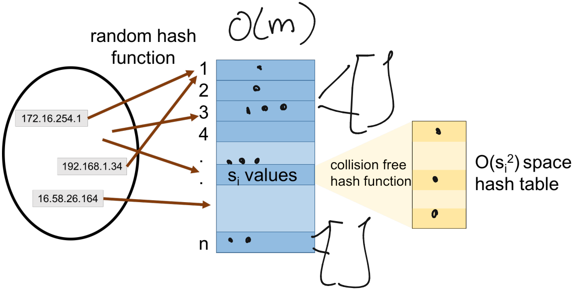

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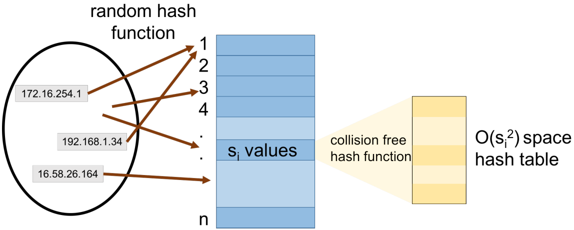

- For each bucket with s_i values, pick a collision free hash function mapping $[s_i] \rightarrow [s_i^2]$.

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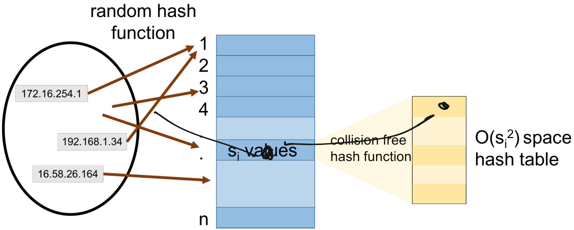

- For each bucket with s_i values, pick a collision free hash function mapping $[s_i] \to [s_i^2]$.
- **Just Showed:** A random function is collision free with probability $\geq \frac{7}{8}$ so can just generate a random hash function and check if it is collision free.

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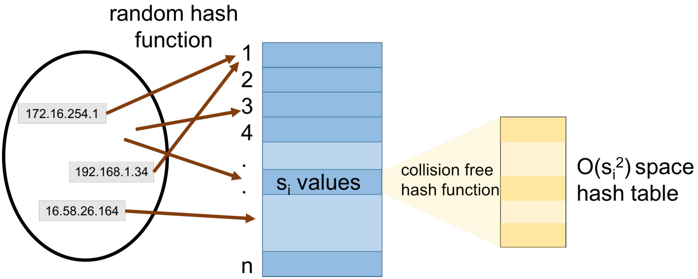

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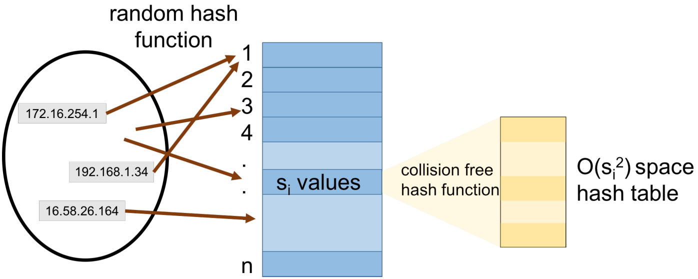

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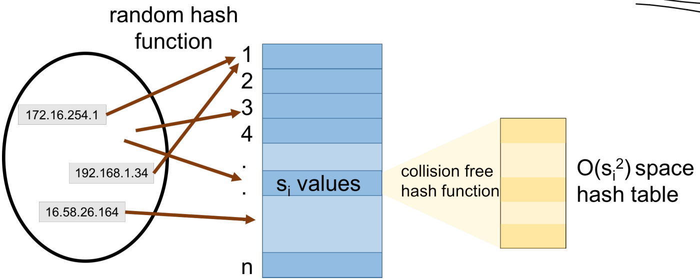
$m = n$



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want to show: $O(n)$

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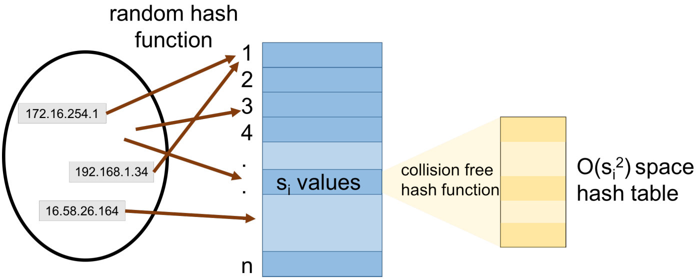

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first table

back up tables

x_j, x_k: stored items, n: hash table size, $\mathbf{h}$: random hash function, $\mathbf{S}$: space usage of two level hashing, $\mathbf{s}_i$: # items stored in hash table at position i.

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$$s_i = \sum_{j=1}^{m} \mathbb{I}_{h(x_j)=i}$$

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$$= \mathbb{E}\left[\sum_{j,k\in[m]} \mathbb{I}_{\mathbf{h}(x_j)=i} \cdot \mathbb{I}_{\mathbf{h}(x_k)=i}\right]$$

$s_i^2 = (I_{1,i} + I_{2,i} + I_{3,i})^2$

$I_{1,i} I_{2,i} + I_{1,i} I_{3,i} + \ldots$

Collisions again!

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$(x+y+z)^2 = x^2 + xy\ldots$

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Si and Sj are not indepent

$$\mathbb{E}[\mathbf{s}_i^2] = \mathbb{E}\left[\left(\sum_{j=1}^{m} \mathbb{I}_{\mathbf{h}(x_j)=i}\right)^2\right]$$

$\mathbb{E}[\mathbb{I}_A] = 1 \cdot Pr(A) + 0 \cdot (1 - Pr(A)) = Pr(A)$

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  $\mathbb{I}_{h(x_j)=i \cap h(x_k)=i}$

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$$= m \cdot \frac{1}{n} + 2 \cdot \binom{m}{2} \cdot \frac{1}{n^2}$$

$(x+y)^2$
$= x^2 + 2xy + y^2$

m terms

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$2\binom{m}{2} = m(m-1)$

x_j, x_k: stored items, m: # stored items, n: hash table size, $\mathbf{h}$: random hash function, $\mathbf{S}$: space usage of two level hashing, $\mathbf{s}_i$: # items stored at pos i.

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n items

$\mathbb{E}[s_i^2] = 2$ in expectation

$$\frac{n}{n} + \frac{n(n-1)}{n^2} \leq 1 + 1 < 2$$

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Total Expected Space Usage: (if we set $n = m$)

$$\mathbb{E}[\mathsf{S}] = n + \sum_{i=1}^{n} \mathbb{E}[\mathsf{s}_i^2]$$

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$$\mathbb{E}[\mathsf{S}] = n + \sum_{i=1}^{n} \mathbb{E}[\mathsf{s}_i^2] \leq n + n \cdot 2 = 3n = 3m.$$

(# items)

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Near optimal space with $O(1)$ query time!

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Efficiently Computable Hash Functions

Beyond Idealized Hash Function

So Far: we have assumed a **fully random hash function** $\mathbf{h}(x)$ with $\Pr[\mathbf{h}(x) = i] = \frac{1}{n}$ for $i \in 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ independent for $x \neq y$.

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- To compute a random hash function we have to store a table of x values and their hash values. Would take at least $O(m)$ space and $O(m)$ query time to look up $\mathbf{h}(x)$ if we hash m values. Making our whole quest for $O(1)$ query time pointless!

x	**h(x)**
x_1	45
x_2	1004
x_3	10
$\vdots$	$\vdots$
x_m	12

Efficiently Computable Hash Functions

What properties did we use of the randomly chosen hash function?

$\hookrightarrow \Pr(h(x_j) = i) = \frac{1}{n}$

$j \neq k \quad \Pr(h(x_j) = h(x_k) = i) = \frac{1}{n^2}$

What properties did we use of the randomly chosen hash function?

Pairwise Independent Hash Function. A random hash function from $\mathbf{h} : U \to [n]$ is pairwise independent if for all $i, j \in [n]$:

$$\Pr[\mathbf{h}(x) = i \cap \mathbf{h}(y) = j] = \frac{1}{n^2}.$$

$x \neq y$

$Pr[h(x) = i \cap h(y) = i] = \frac{1}{n^2}$

x, y, z

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$\Pr\left(h(x_j) = i \cap h(x_k) = i \cap h(x_l) = i \right) \neq \frac{1}{n^3}$

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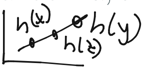
Efficient Implementation: Let p be a prime with $p \geq |U|$. Choose random $\mathbf{a}, \mathbf{b} \in [p]$ with $\mathbf{a} \neq 0$. Represent x as an integer and let



$$\mathbf{h}(x) = (\mathbf{a}x + \mathbf{b} \mod p) \mod n.$$

Another common requirement for a hash function:

2-Universal Hash Function (low collision probability). A random hash function from $\mathbf{h} : U \to [n]$ is two universal if:

$x \neq y$

$$\Pr[\mathbf{h}(x) = \mathbf{h}(y)] \leq \frac{1}{n}.$$

$$\Pr[h(x) = h(y)] = \frac{1}{n}$$

Universal Hashing

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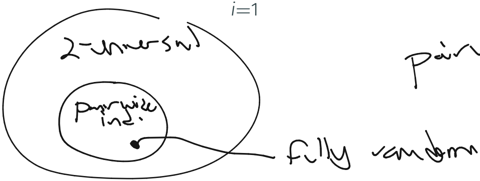
pairwise ind $\Rightarrow$ 2 universal



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Remember: A fully random hash function is both 2-universal and pairwise independent. But it is not efficiently implementable.

Universal Hashing

Another common requirement for a hash function:

$h(1) = 1$
$h(2) = 2$
$h(i) =$ random in $1, \ldots, n$ for $i > 2$

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Exercise 2: Rework the two-level hashing proof to show that 2-universality is in fact all that is needed.

Questions on Hash Tables?

Next Step

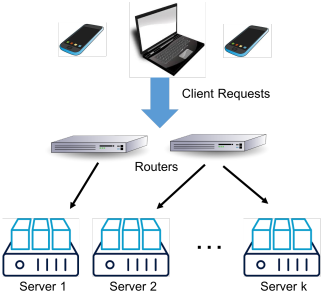
1. We'll consider an application where our toolkit of linearity of expectation + Markov's inequality doesn't give much.

2. Then we'll show how a simple twist on Markov's (Chebyshev's inequality) can give a much stronger result.

Hashing for Load Balancing

Another Application

Randomized Load Balancing:



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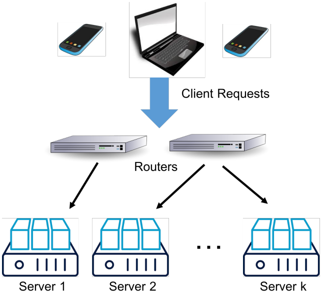

Simple Model: n requests randomly assigned to k servers. How many requests must each server handle?

- Often assignment is done via a random hash function. Why?

Weakness of Markov's

$\mathbb{E}[\mathsf{R}_i] =$

n: total number of requests, k: number of servers randomly assigned requests, R_i: number of requests assigned to server i.

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$$\Pr[\mathsf{R}_i \geq 2\mathbb{E}[\mathsf{R}_i]] \leq \frac{\mathbb{E}[\mathsf{R}_i]}{2\mathbb{E}[\mathsf{R}_i]} = \frac{1}{2}.$$

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Not great...half the servers may be overloaded.

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Chebyshev's inequality

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Chebyshev's inequality:

$$\Pr(|X - \mathbb{E}[X]| \geq t) \leq \frac{\mathrm{Var}[X]}{t^2}.$$

(by plugging in the random variable $X - \mathbb{E}[X]$)