

COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2026.

Lecture 11

Logistics

- Midterm solutions are posted. Grades in progress.
- Online quiz due Monday morning.
- In-class quiz on Monday.

Summary

- Nearest neighbor search/vector databases.
- Solution via locality sensitive hashing.

Fast Similarity Search

Have a database of items – e.g., documents, images, audio clips, etc. Often they have transformed into ‘embeddings’ – i.e., representations as vectors or sets.

Define a similarity metric over these items – e.g., cosine similarity (dot product) if they are represented as vectors, or Jaccard similarity if represented as sets.

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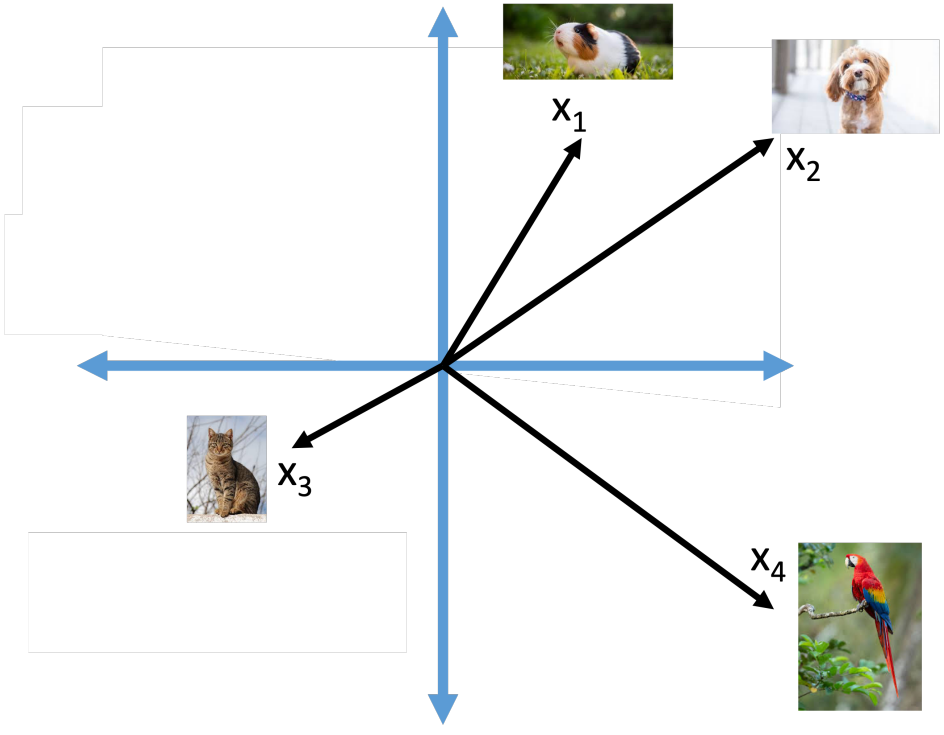
- **Near Neighbor Search:** Given a query item q, find if it has high similarity to any database item. $\Omega(n)$ time with a linear scan.
- **All-pairs Similarity Search:** Have n different query vectors and want to find all pairs with high similarity. $\Omega(n^2)$ time if we check all pairs explicitly.

Difficulty is that q almost never has an exact match in the database – if it did we could solve search in $O(1)$ time using a hash table.

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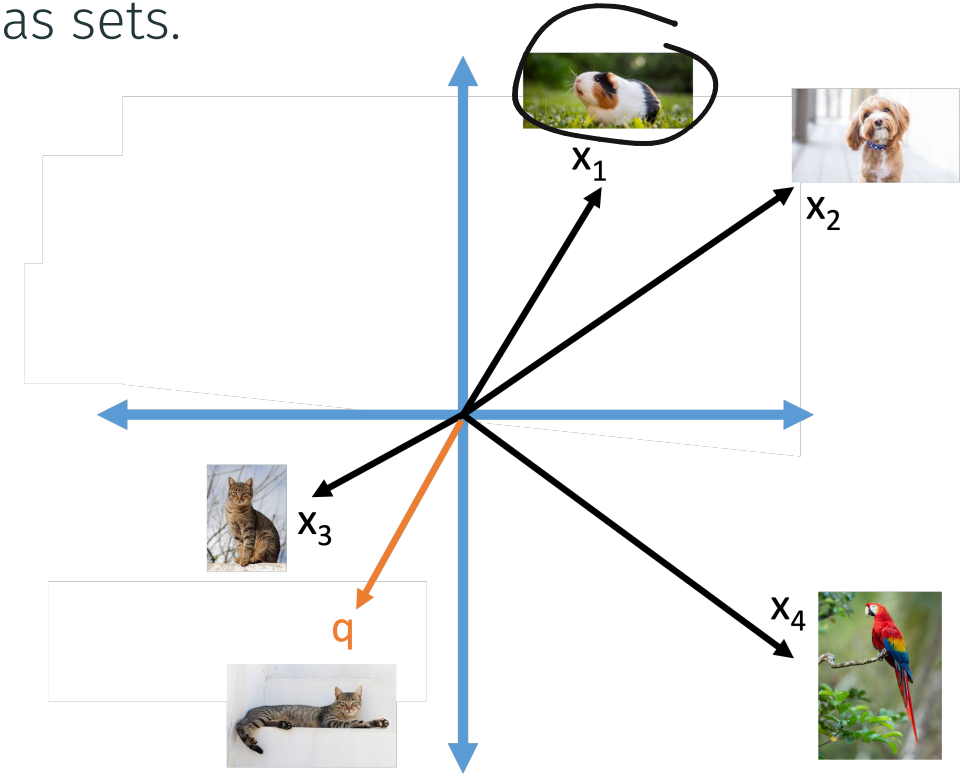
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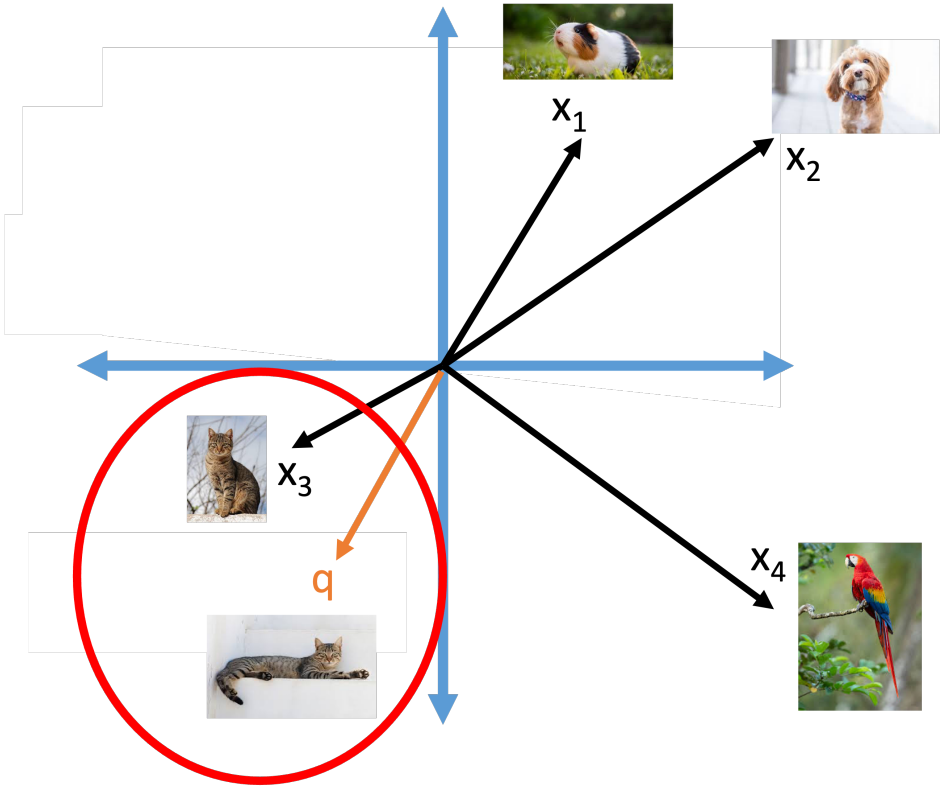
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Applications of Fast Similarity Search

Huge number of applications including:

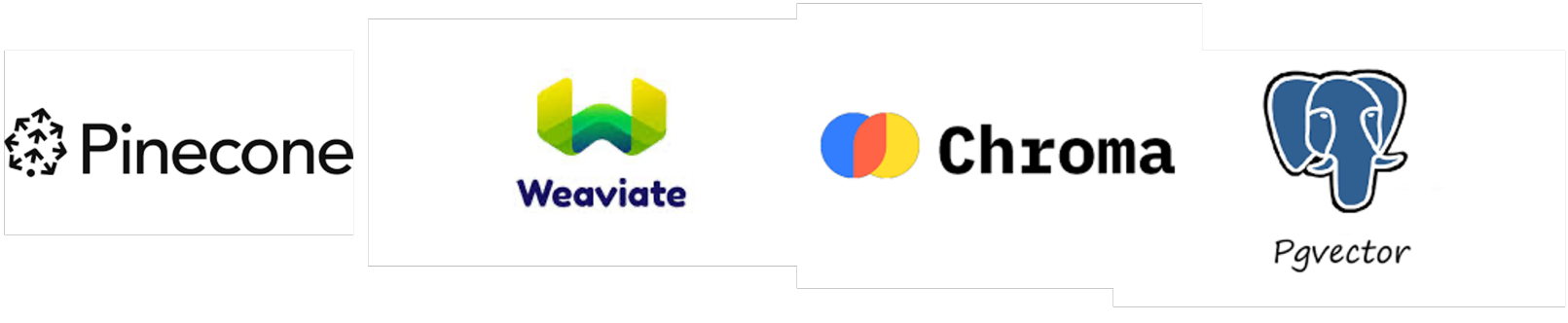
- Reverse image search (e.g., Google)
- Audio search (e.g., Shazam)
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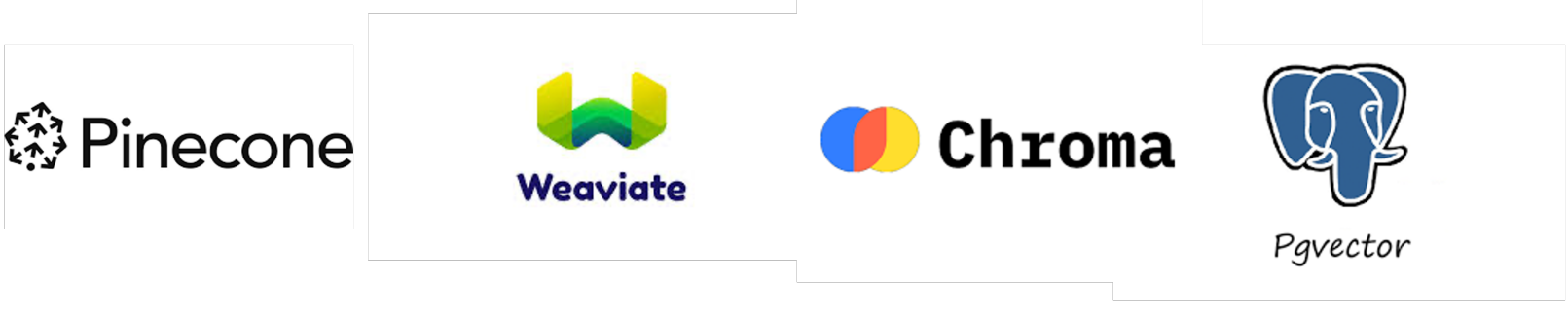


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This area has blown up in the last year or two due to the importance of RAG in particular.

Jaccard Similarity

The most popular similarity metrics in practice are cosine similarity (inner product) or ℓ_2 distance between real-valued vectors.

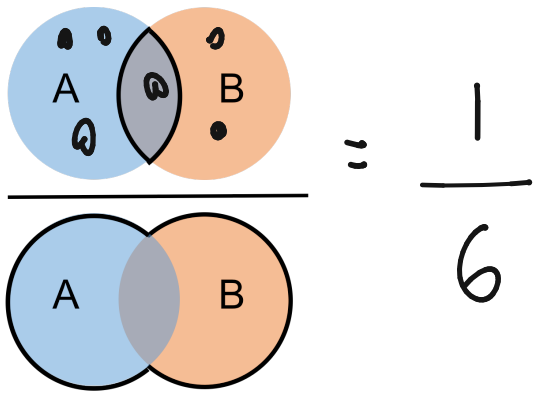
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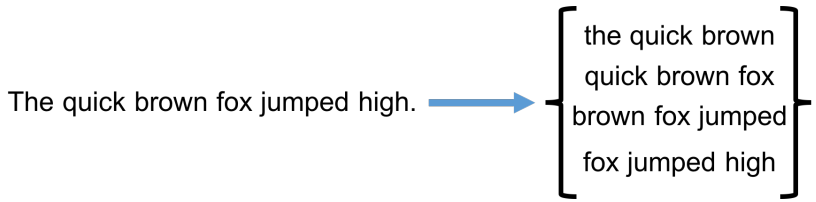


Also a natural measure for similarity between bit strings – interpret an n bit string as a set, containing the elements corresponding the positions of its ones. $J(x, y) = \frac{\#\text{ shared ones}}{\text{total ones}}$.

Application: Document Similarity

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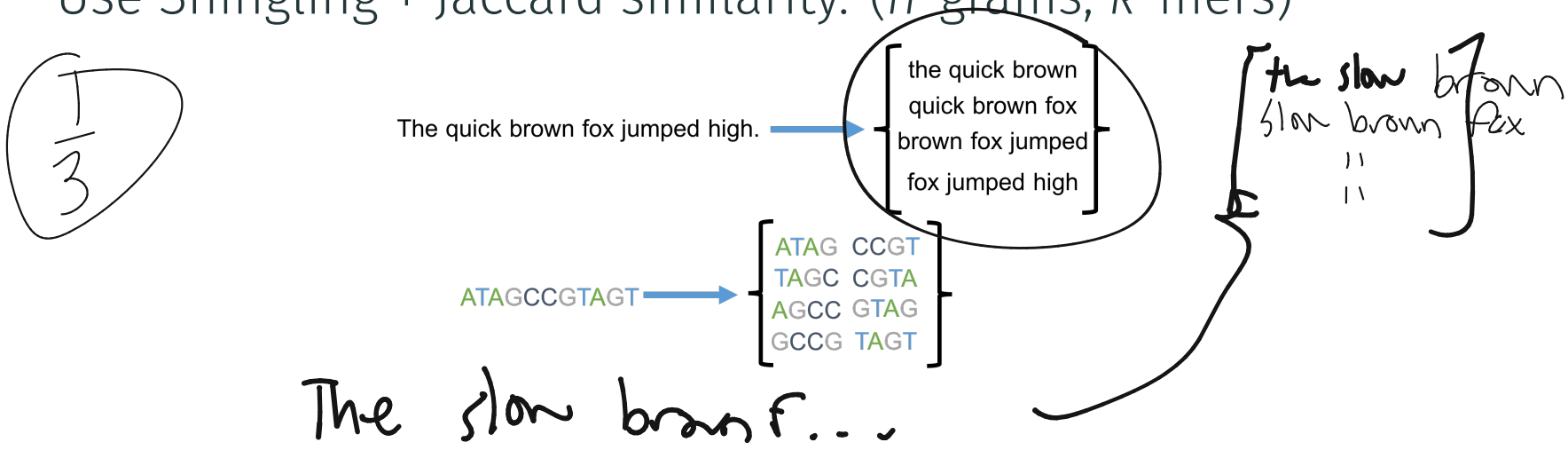
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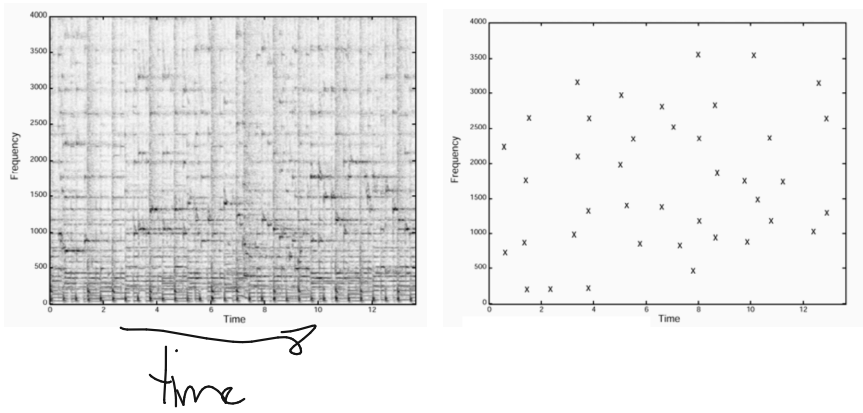
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Application: Audio Search

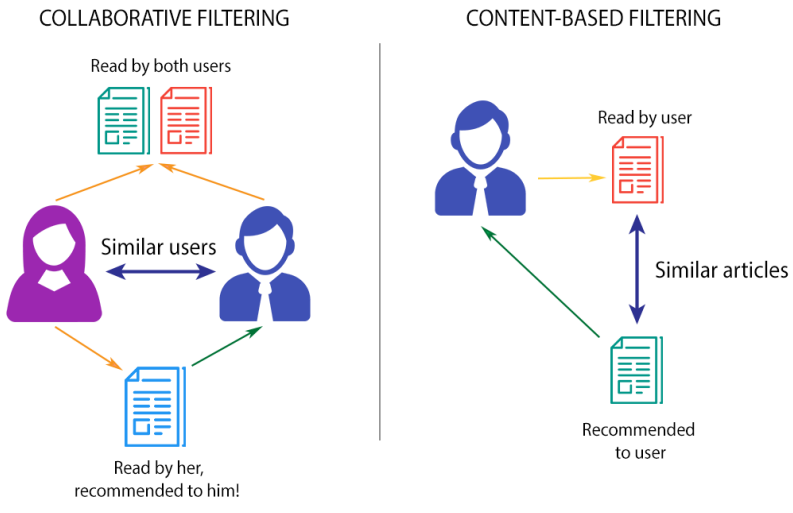
Audio Fingerprinting:

- E.g., in audio search (Shazam), Earthquake detection.
- Represent sound clip via a binary 'fingerprint' then compare with Jaccard similarity.



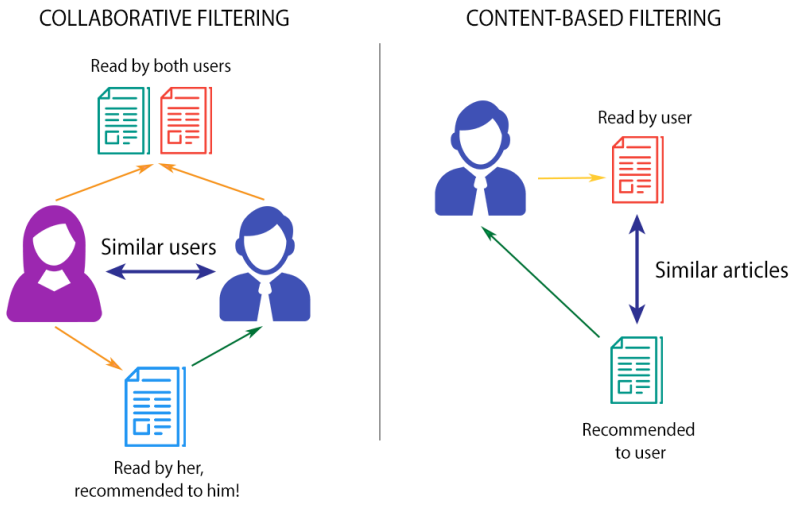
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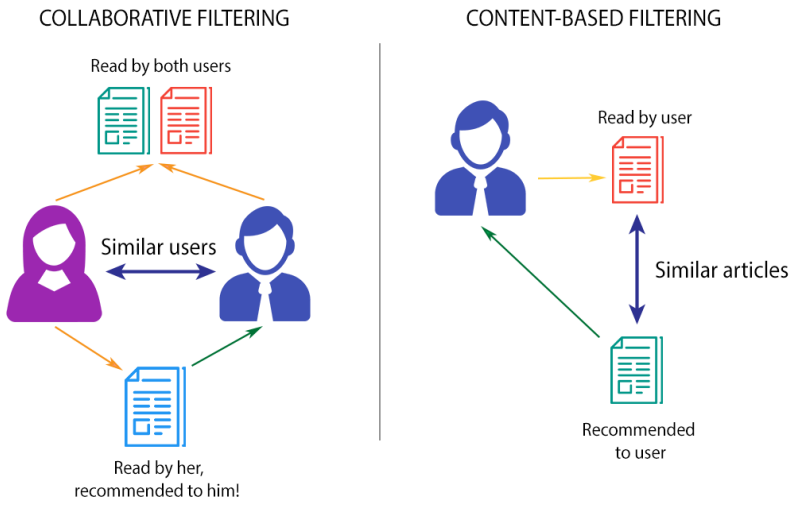
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- Netflix: look at sets of movies watched. Amazon: look at products purchased, etc.

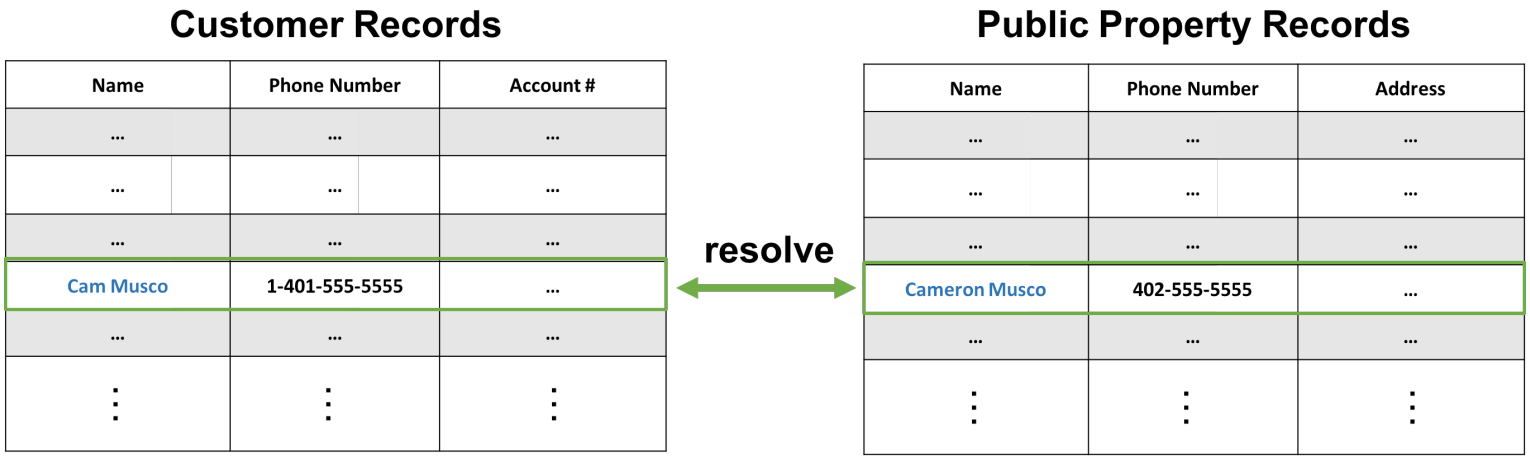
Application: Entity Resolution

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See Section 3.8.2 of *Mining Massive Datasets* for a discussion of a real world example involving 1 million customers. Naively this would be $\binom{1000000}{2} \approx 500$ billion pairs of customers to check!

Fast Similarity Search

Recall: Our goal is to design much faster algorithms for the following problems:

- **Near Neighbor Search:** Given a query item q, find if it has high similarity to any database item. $\Omega(n)$ time with a linear scan.
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We will focus on LSH today, and potentially cover other methods in later lectures.

Search with Jaccard Similarity

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Want Fast Implementations For:

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- **All-pairs Similarity Search:** Have n different sets/bit strings and want to find all pairs with high Jaccard similarity. $\Omega(n^2)$ time if we check all pairs explicitly.

Will speed up via randomized locality sensitive hashing.

Locality Sensitive Hashing

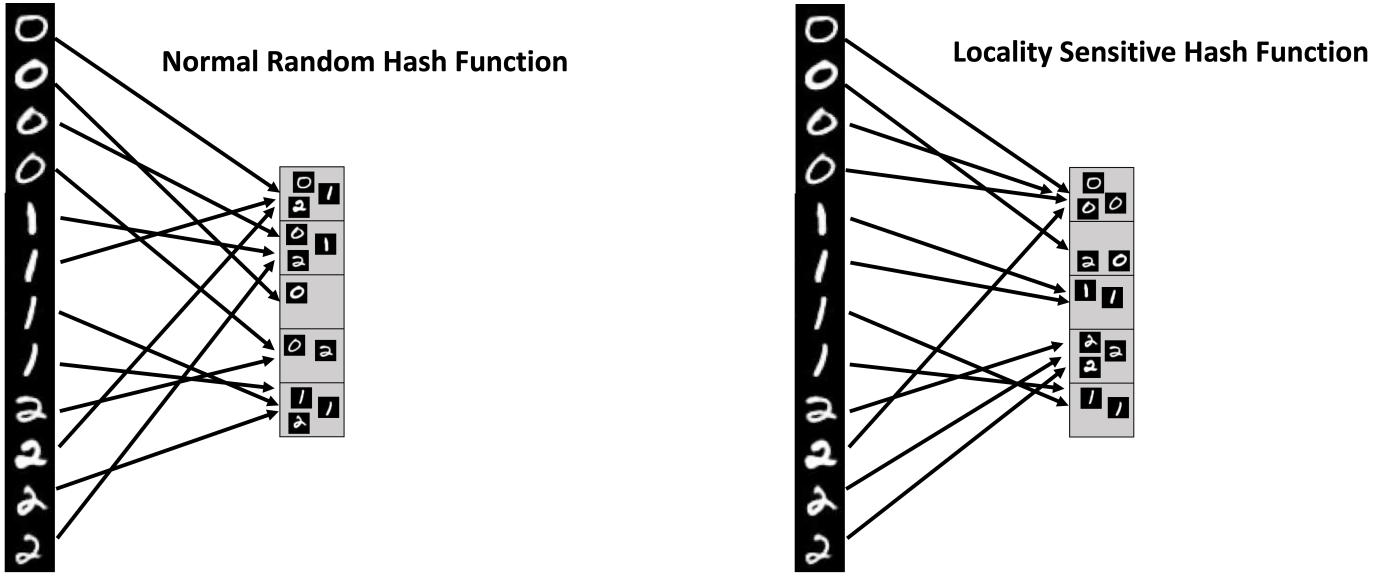
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- Design a hash function where the collision probability is higher when two inputs are more similar (can design different functions for different similarity metrics.)

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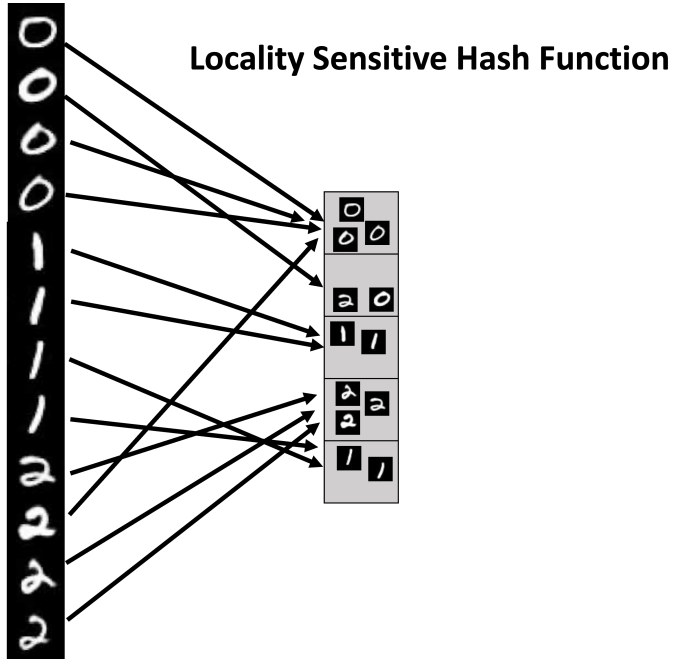
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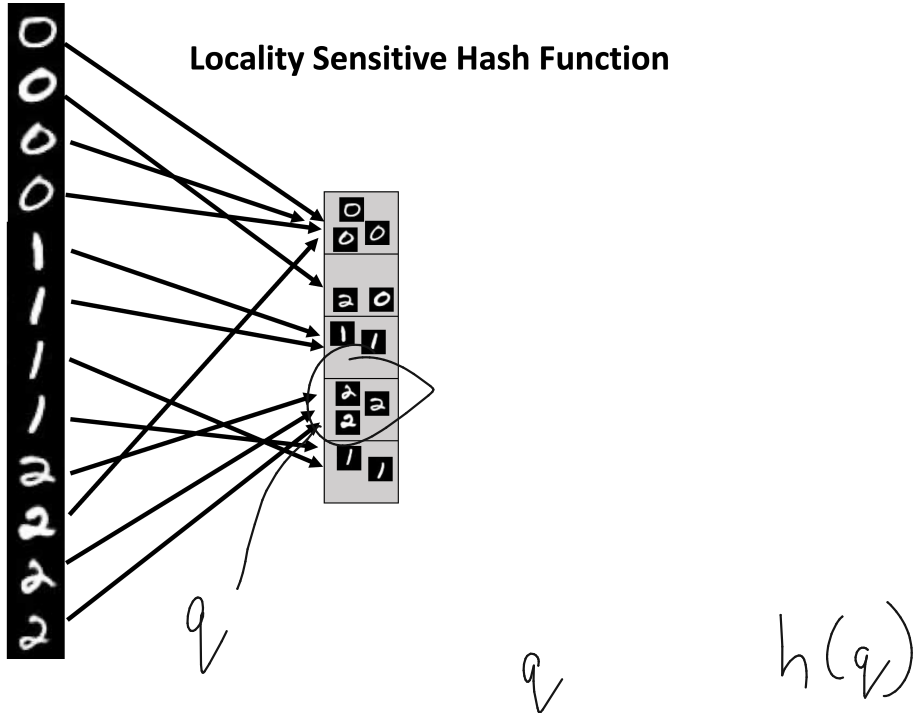
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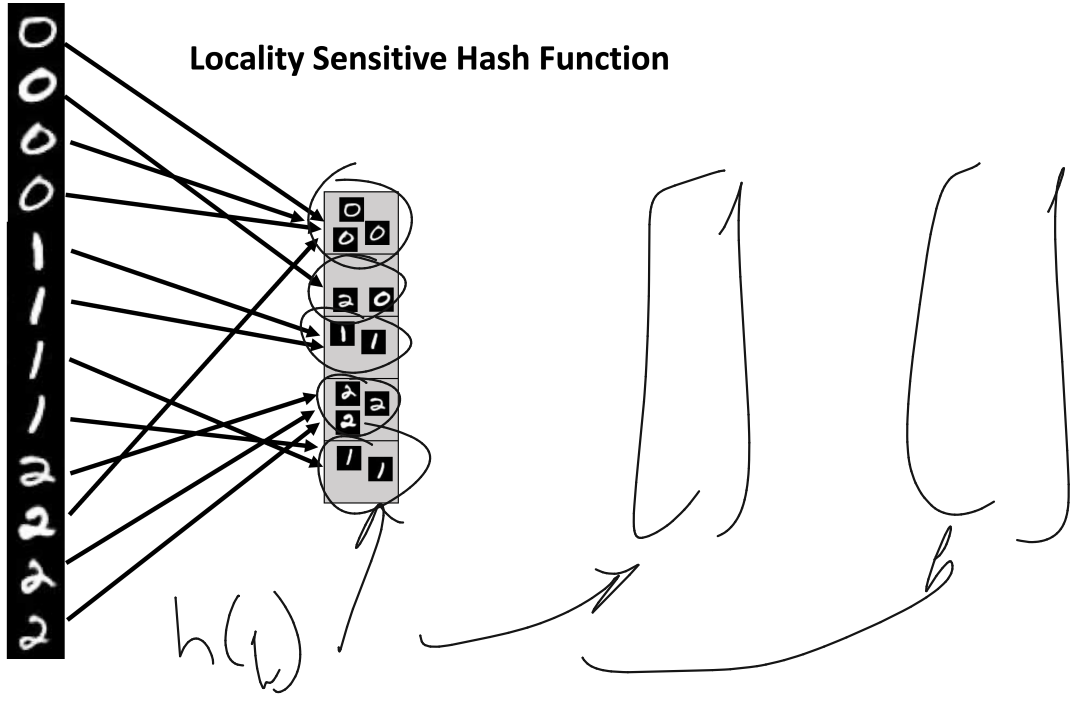
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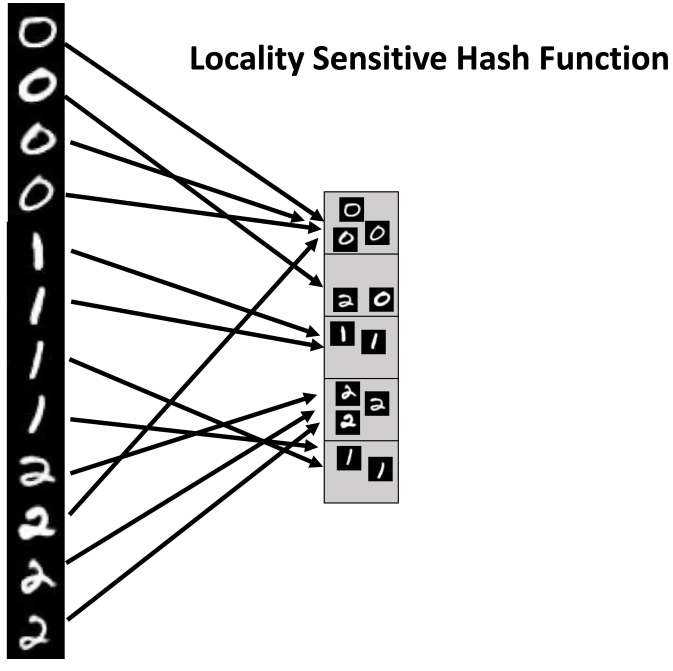
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~~Next Class: How can we design such a hash function?~~

MinHashing

An Example: Locality sensitive hashing for Jaccard similarity.

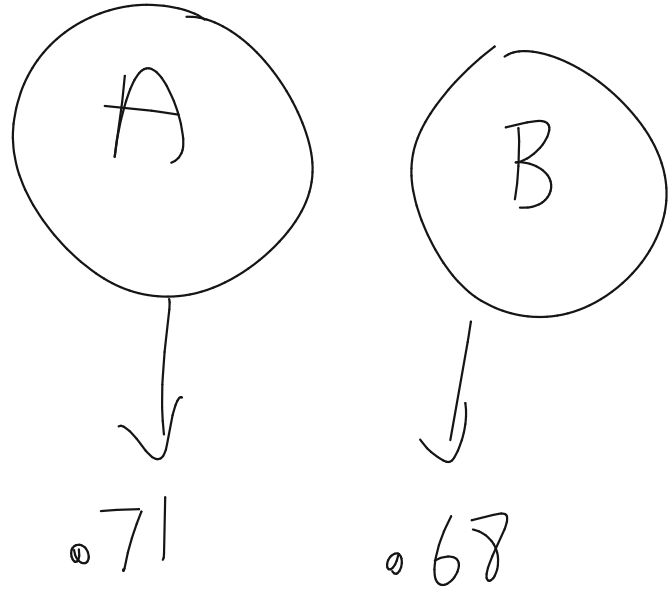
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Strategy: Use random hashing to map each set to a single hash value. The probably that two sets have colliding hash values will be proportional to their Jaccard similarity.

MinHash(A): [Andrei Broder, 1997 at Altavista]

- Let $\mathsf{h} : U \to [0, 1]$ be a random hash function
- $\mathsf{s} := 1$
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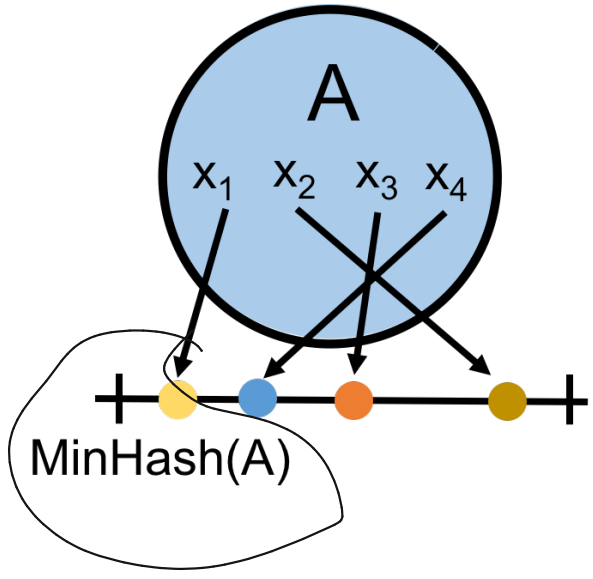
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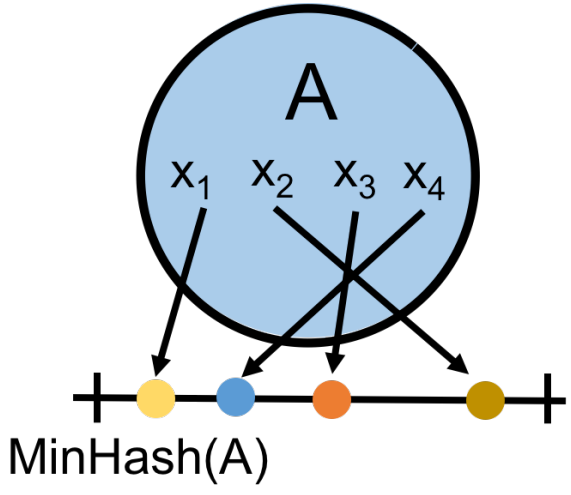
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Identical to our distinct elements sketch!

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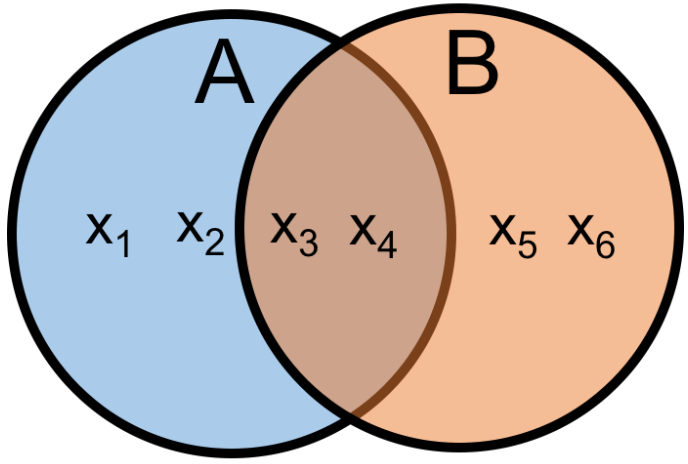
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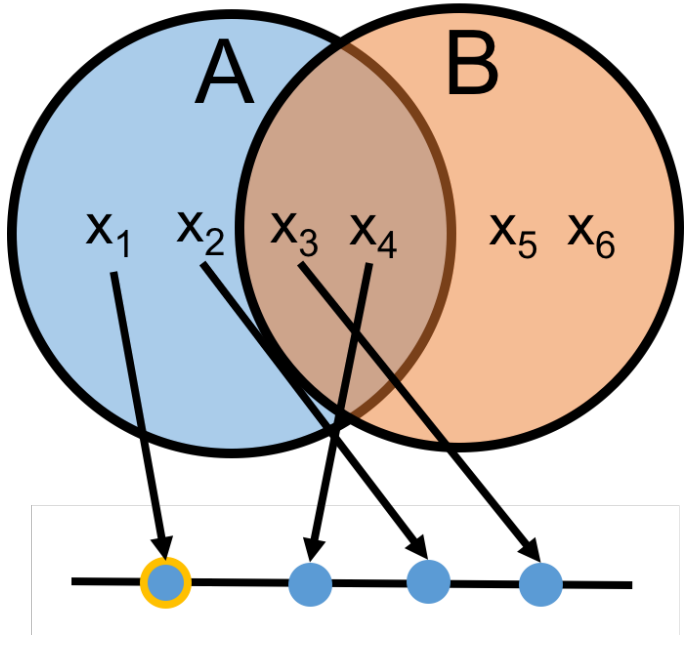


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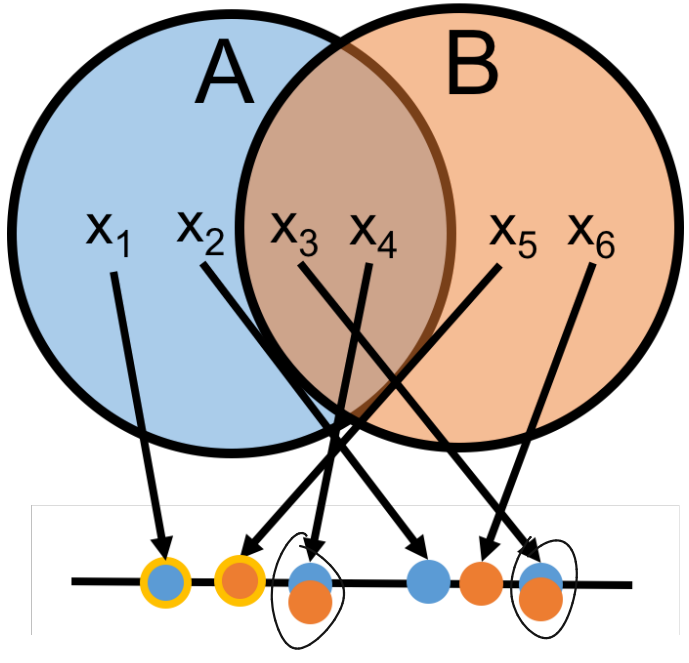
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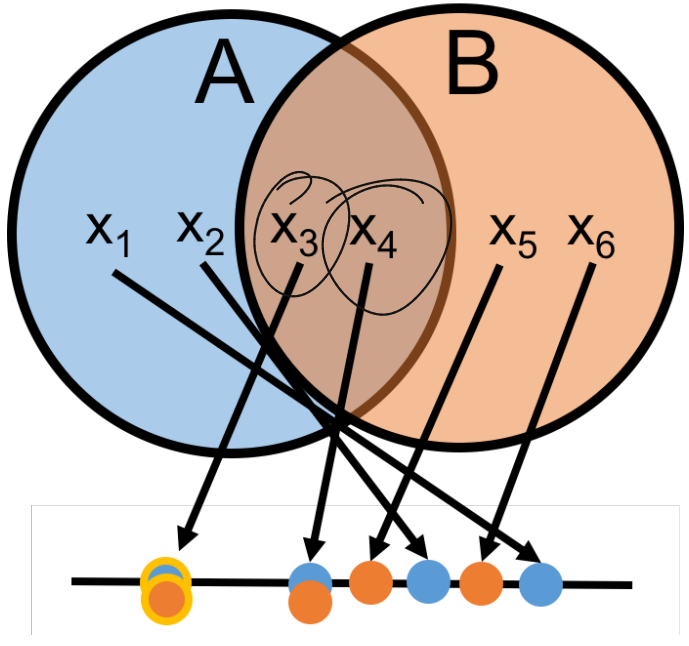


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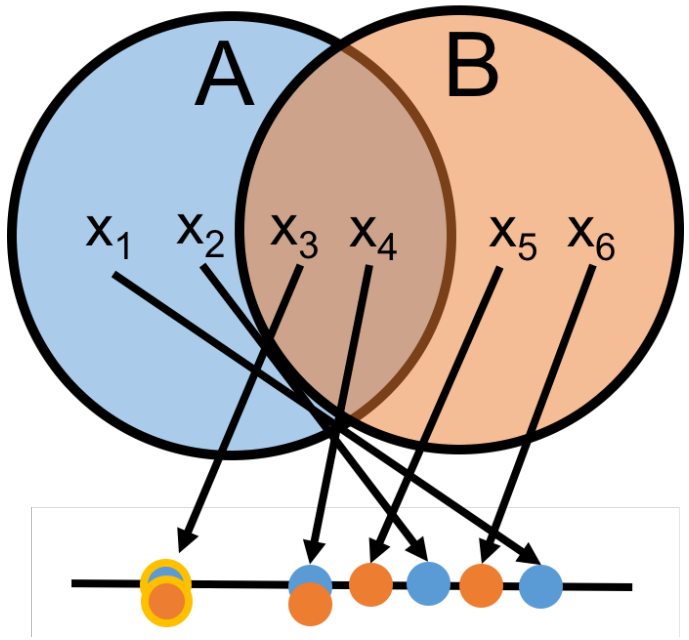


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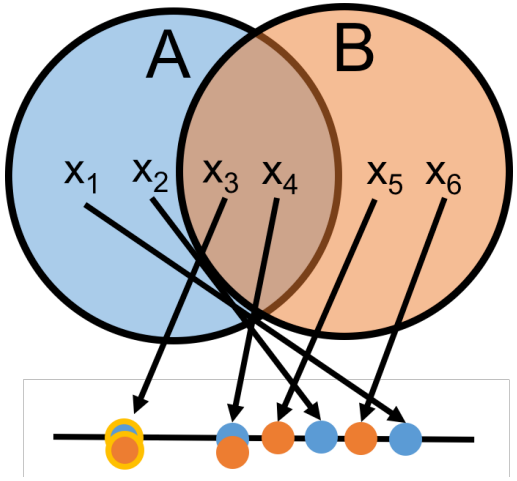
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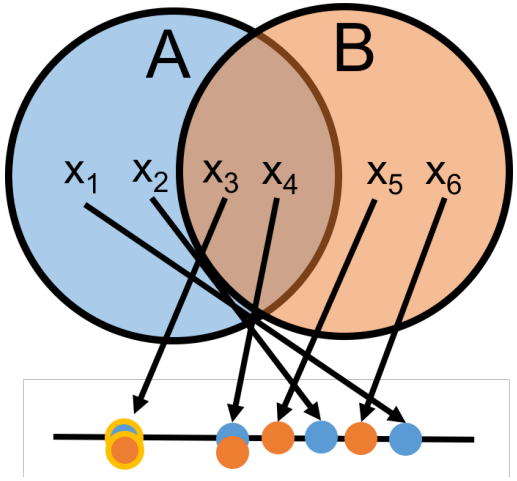
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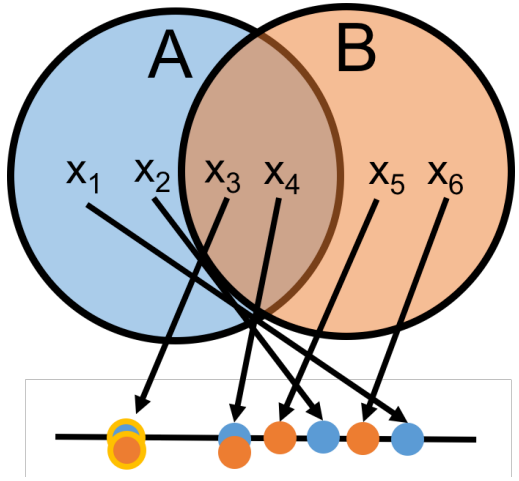


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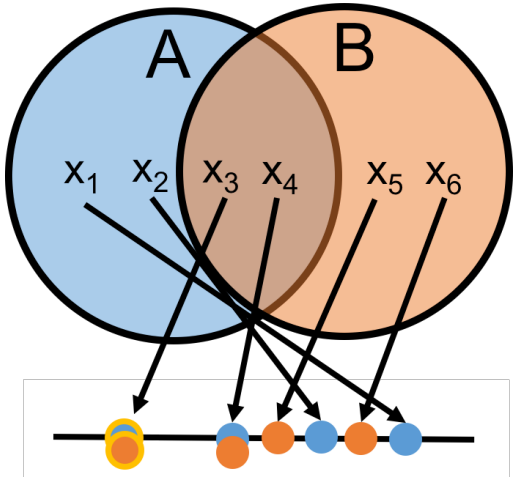


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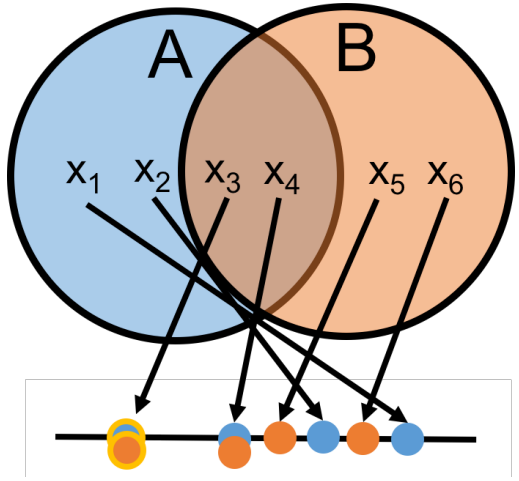
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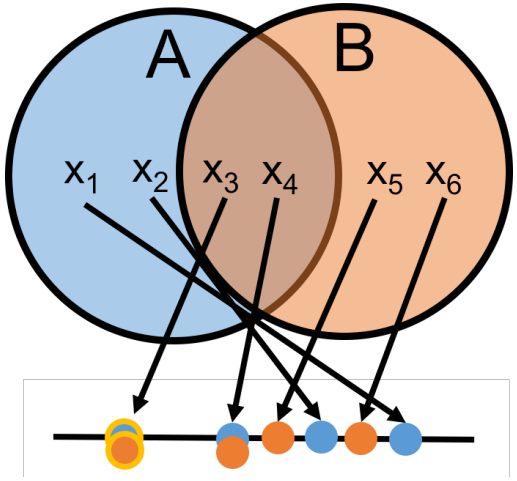


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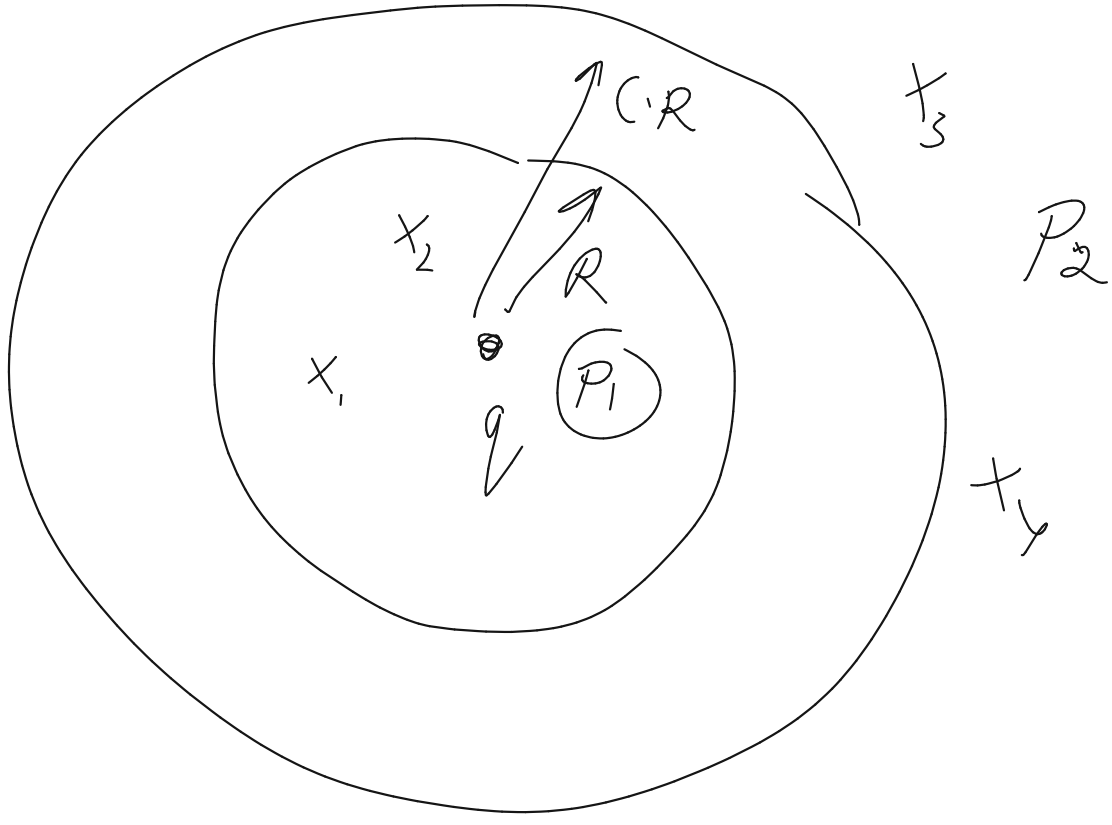


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Locality sensitive: the higher $J(A, B)$ is, the more likely $MinHash(A), MinHash(B)$ are to collide.

Similarity Search with MinHash

Goal: Given a document y, identify all documents x in a database with Jaccard similarity (of their shingle sets) $J(x, y) \geq 1/2$.



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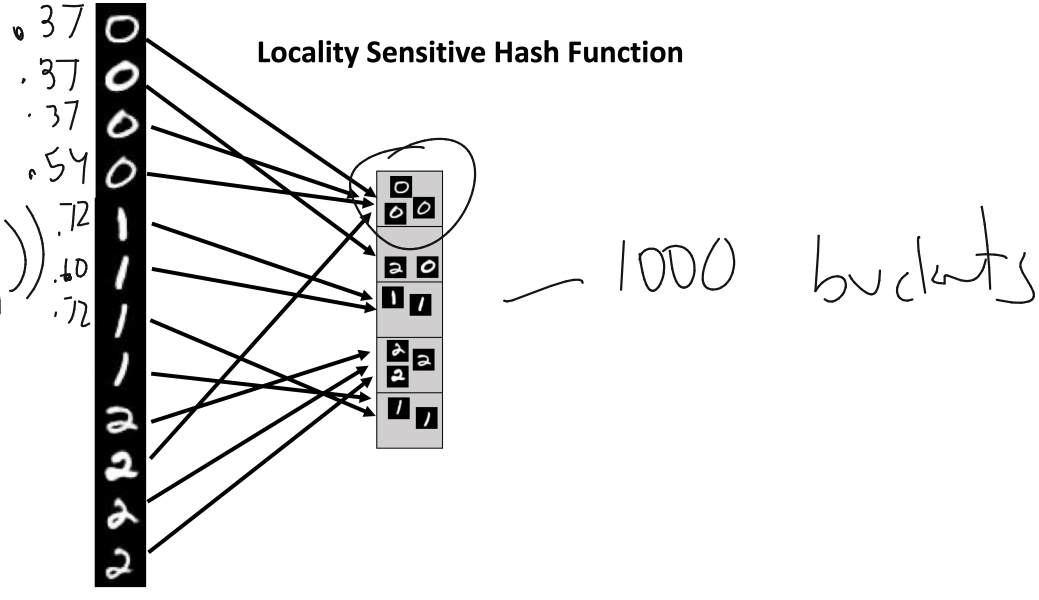
Our Approach:

- Create a hash table of size m, choose a random hash function $g : [0, 1] \to [m]$, and insert every item x into bucket $g(MinHash(x))$. Search for items similar to y in bucket $g(MinHash(y))$.

$mH(x) = mH(y)$

$\Rightarrow g(mH(x)) = g(mH(y))$

$\Rightarrow$ same bucket



~ 1000 buckets

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With a simple use of MinHash, we miss a match x with $J(x, y) = 1/2$ with probability 1/2. How can we reduce this false negative rate?

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↳ space complexity $O(nt)$

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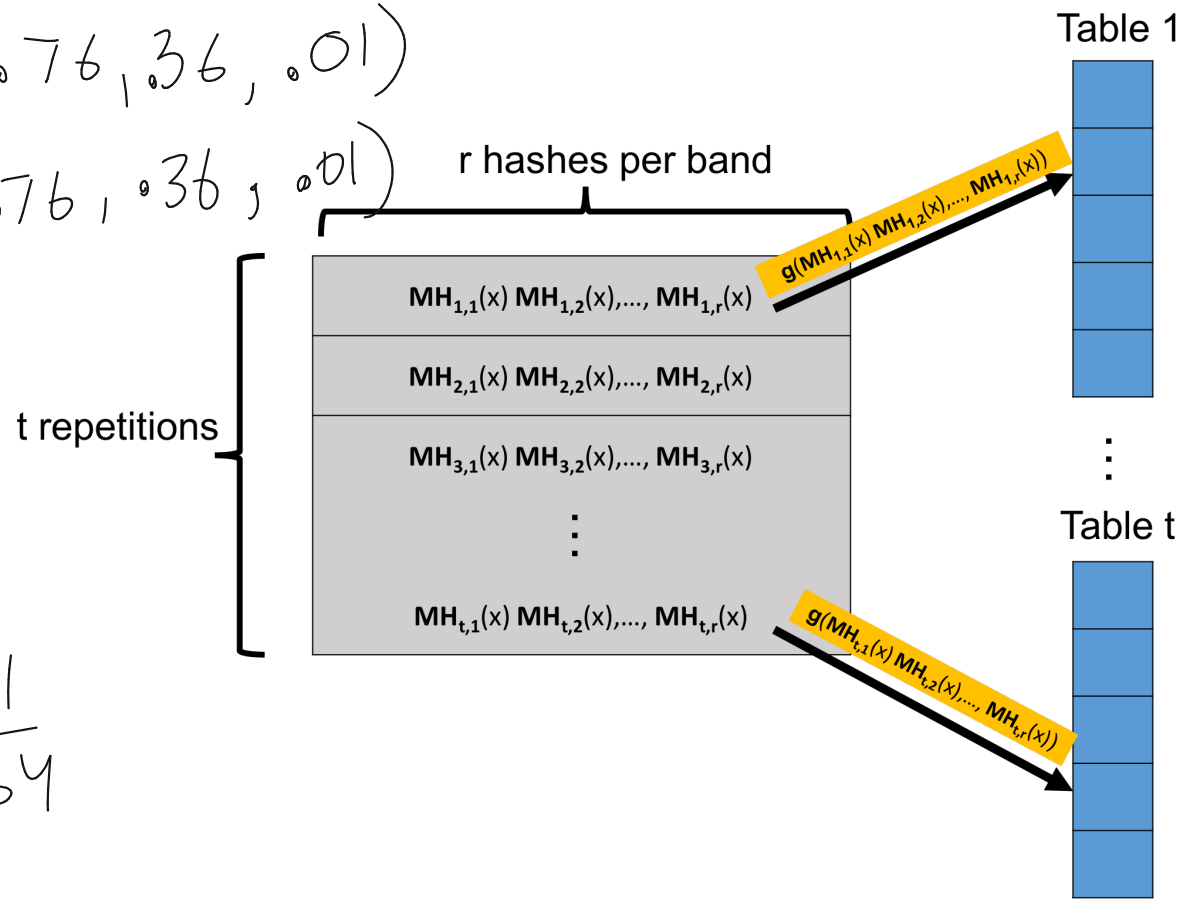
Potential for a lot of false positives! Slows down search time.

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We want to balance a small probability of false negatives (a high hit rate) with a small probability of false positives (a small query time.)

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Create *t* hash tables. Each is indexed into not with a single MinHash value, but with *r* values, appended together. A length *r* signature.

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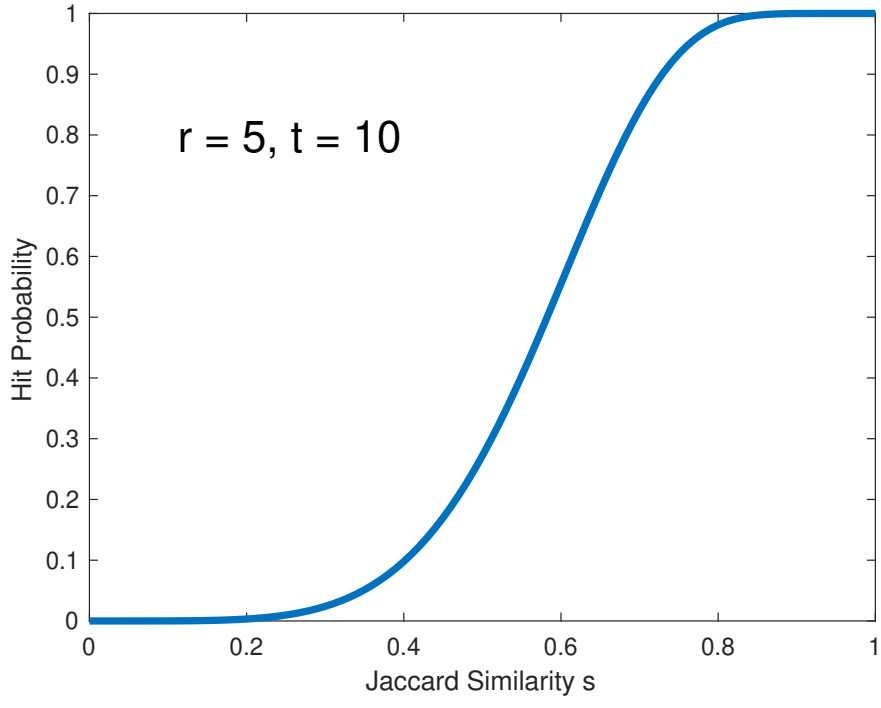
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The s-curve

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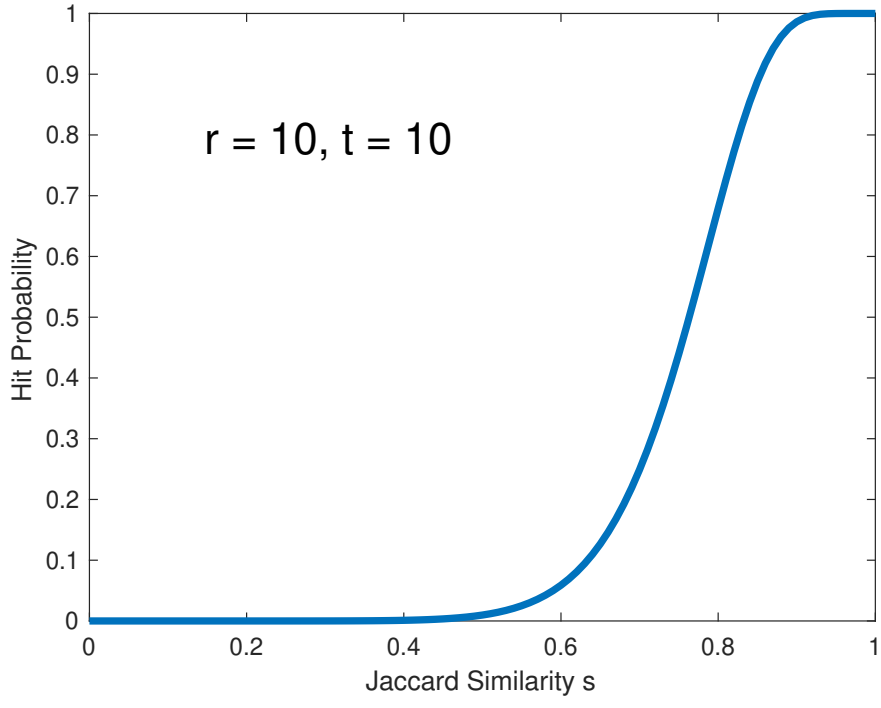
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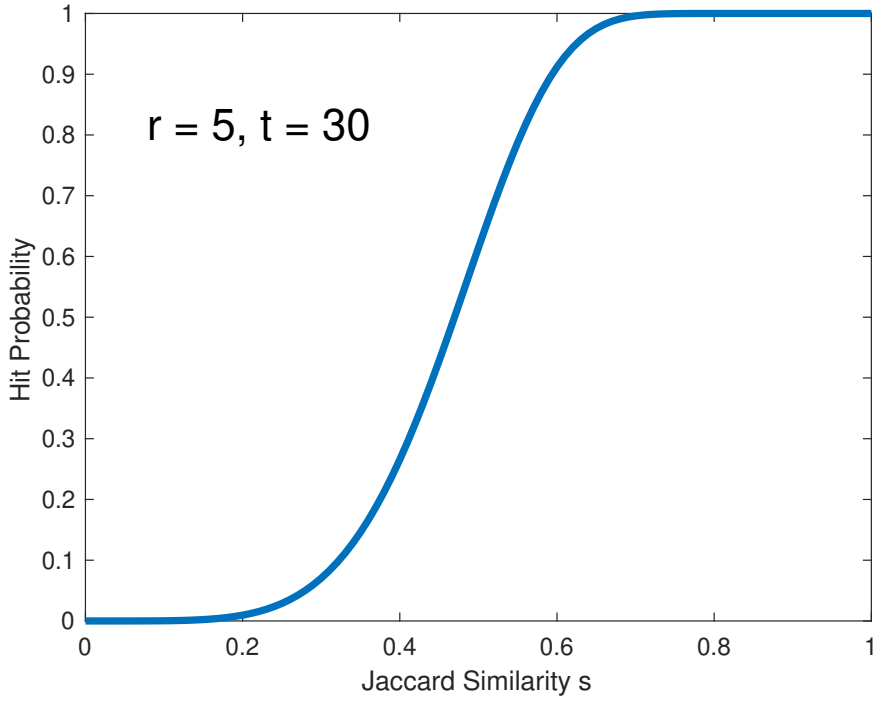
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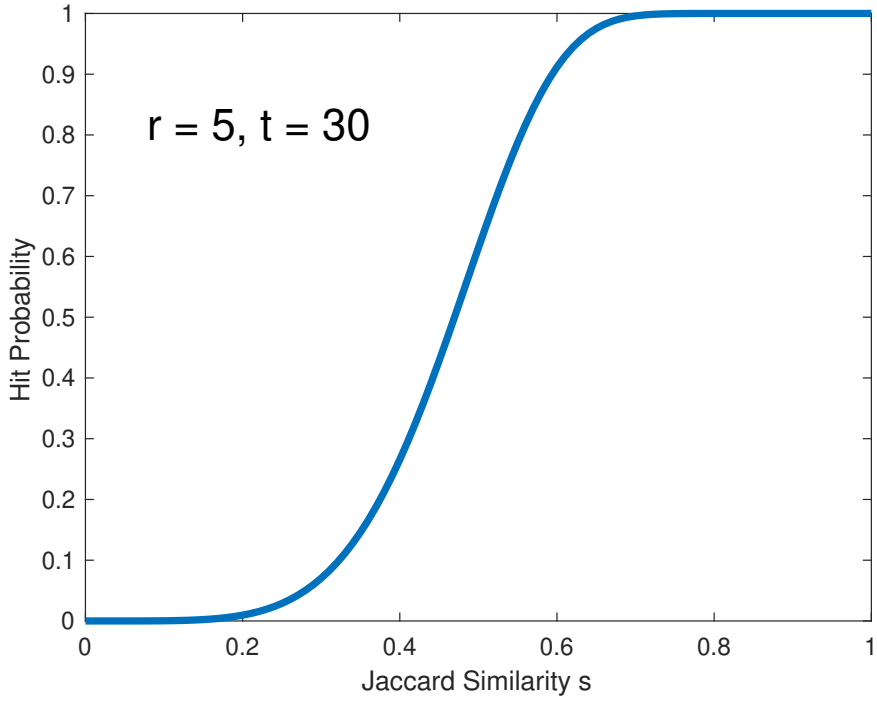
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r and t are tuned depending on application.

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- LSH schemes exist for many similarity/distance measures: hamming distance, cosine similarity, etc.

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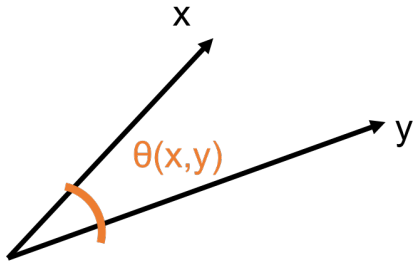
Repetition and s-curve tuning can be used for fast similarity search with any similarity metric, given a locality sensitive hash function for that metric.

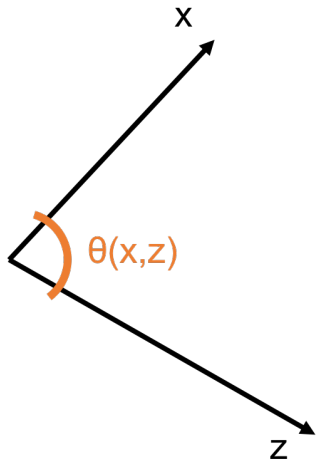
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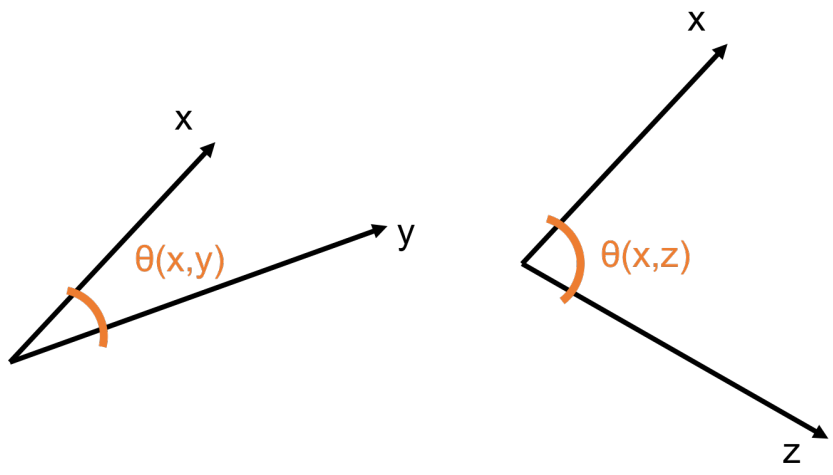


$$\frac{\langle x, y \rangle}{\|x\| \cdot \|y\|}$$

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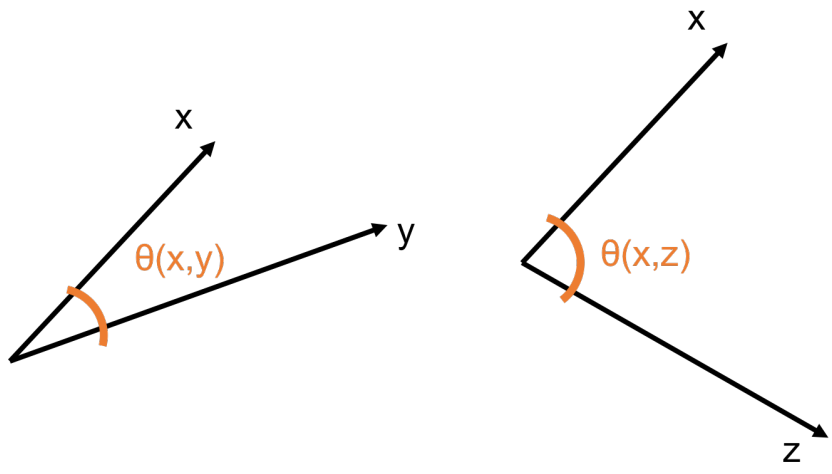


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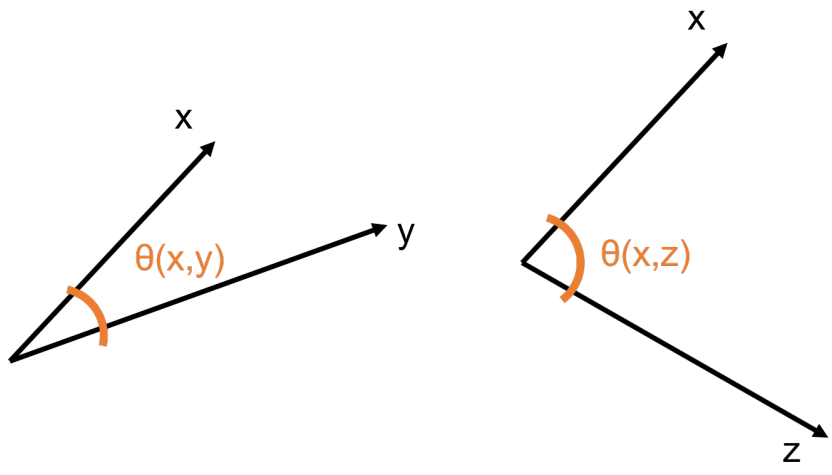
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- $\cos(\theta(x, y)) = 1$ when $\theta(x, y) = 0°$ and $\cos(\theta(x, y)) = 0$ when $\theta(x, y) = 90°$, and $\cos(\theta(x, y)) = -1$ when $\theta(x, y) = 180°$

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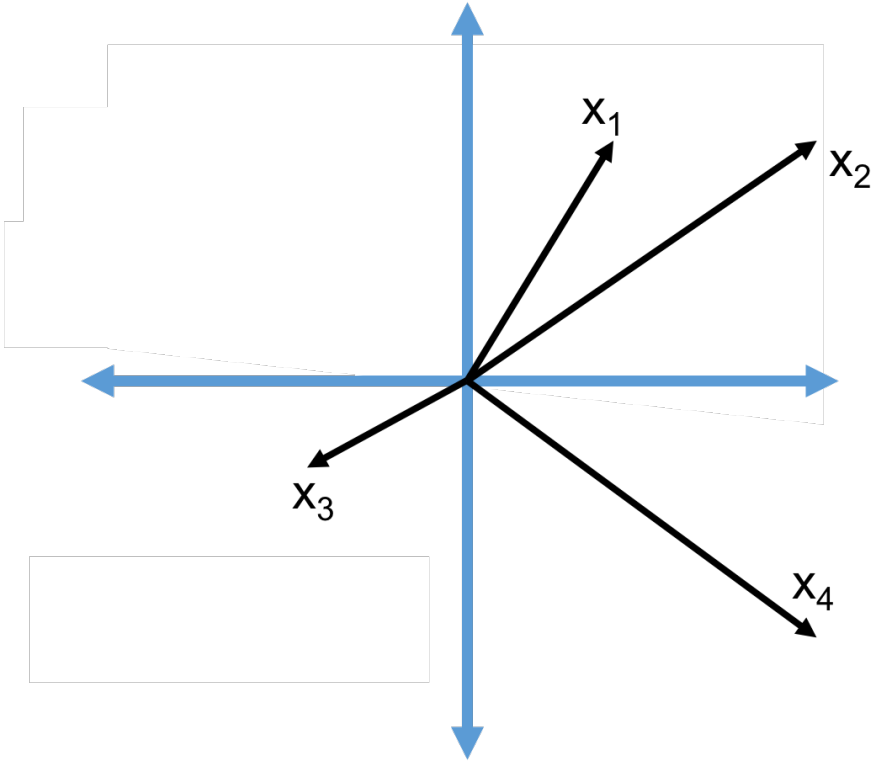
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SimHash for Cosine Similarity

SimHash Algorithm: LSH for cosine similarity.

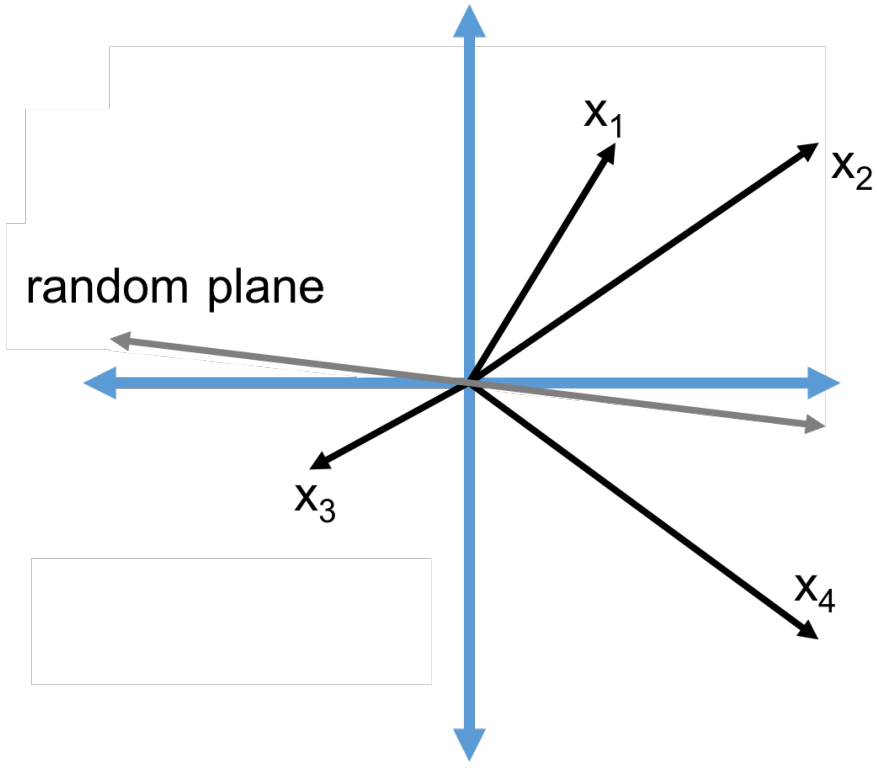
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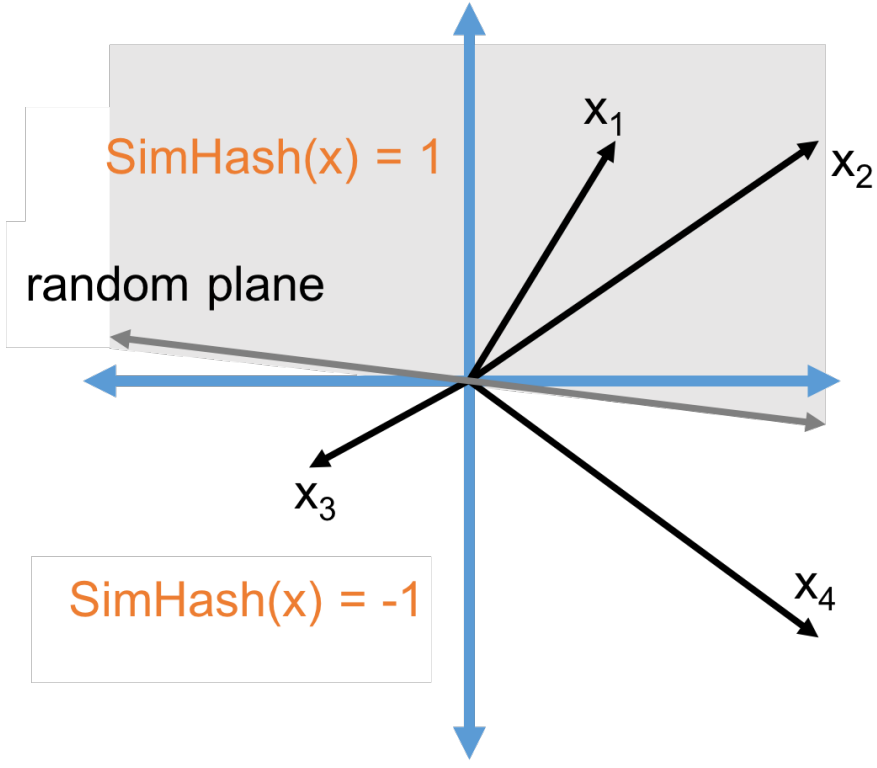


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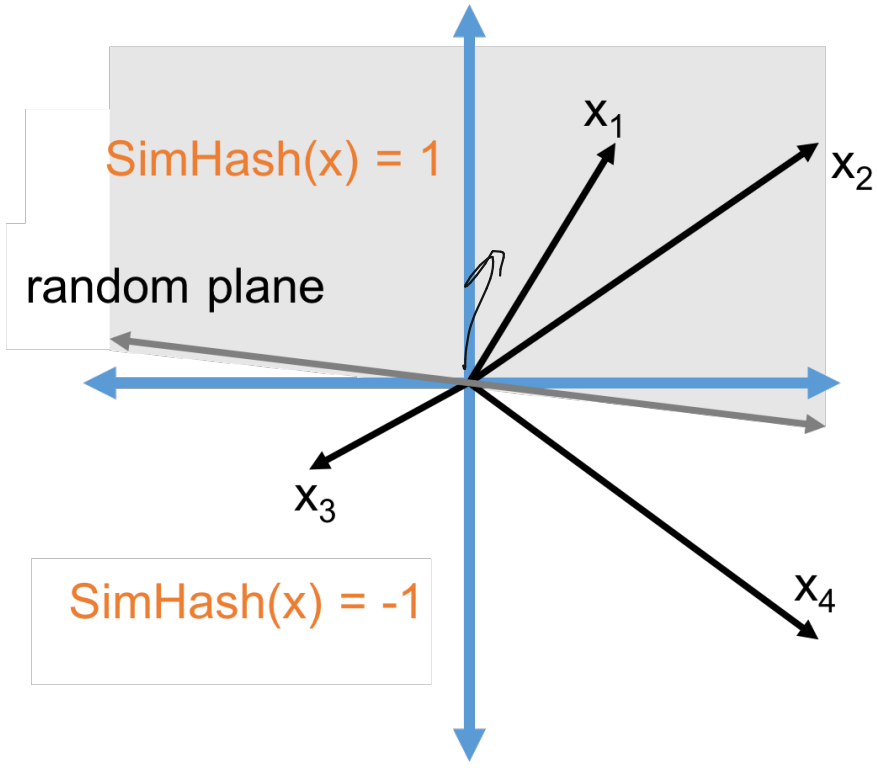
$\mathbb{R}^d \to \{-1, 1\}$

$\{(-1, 1, -1)\}$



SimHash for Cosine Similarity

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$SimHash(x) = \text{sign}(\langle x, t \rangle)$ for a random vector t.

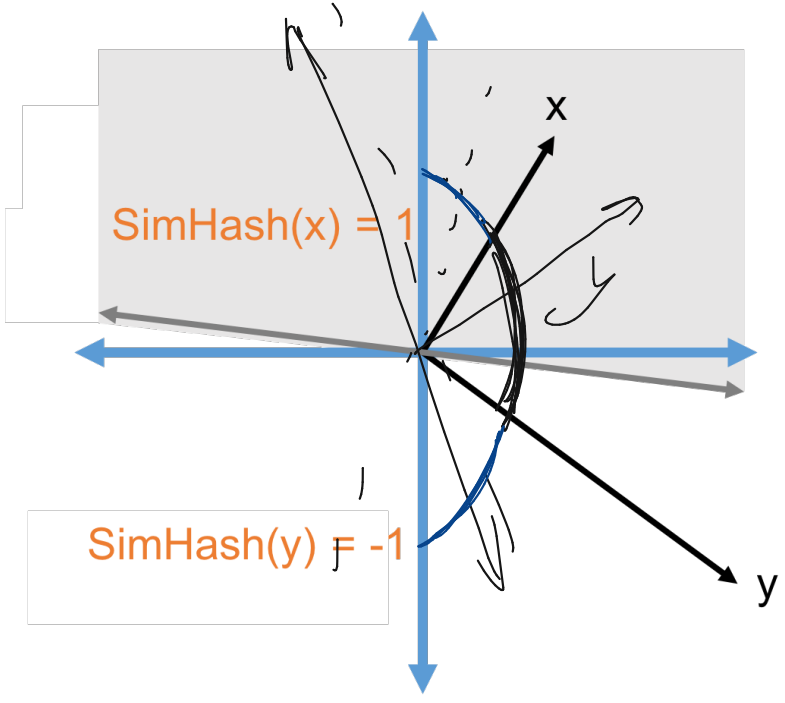
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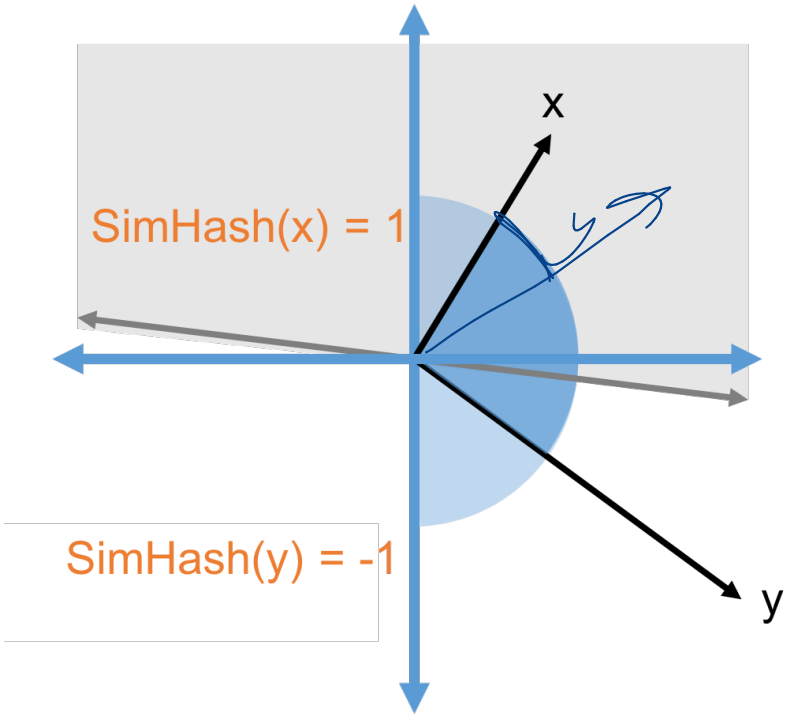
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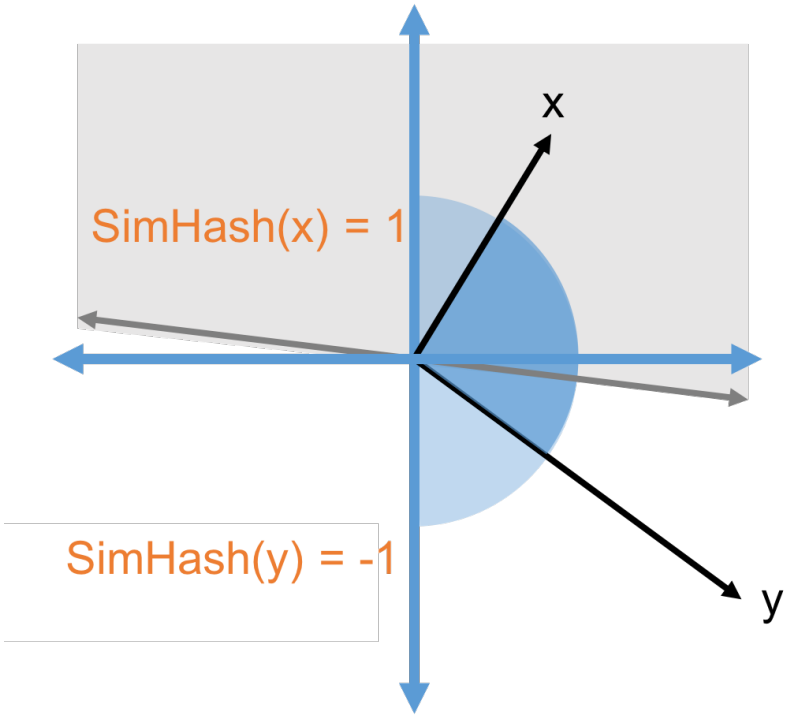
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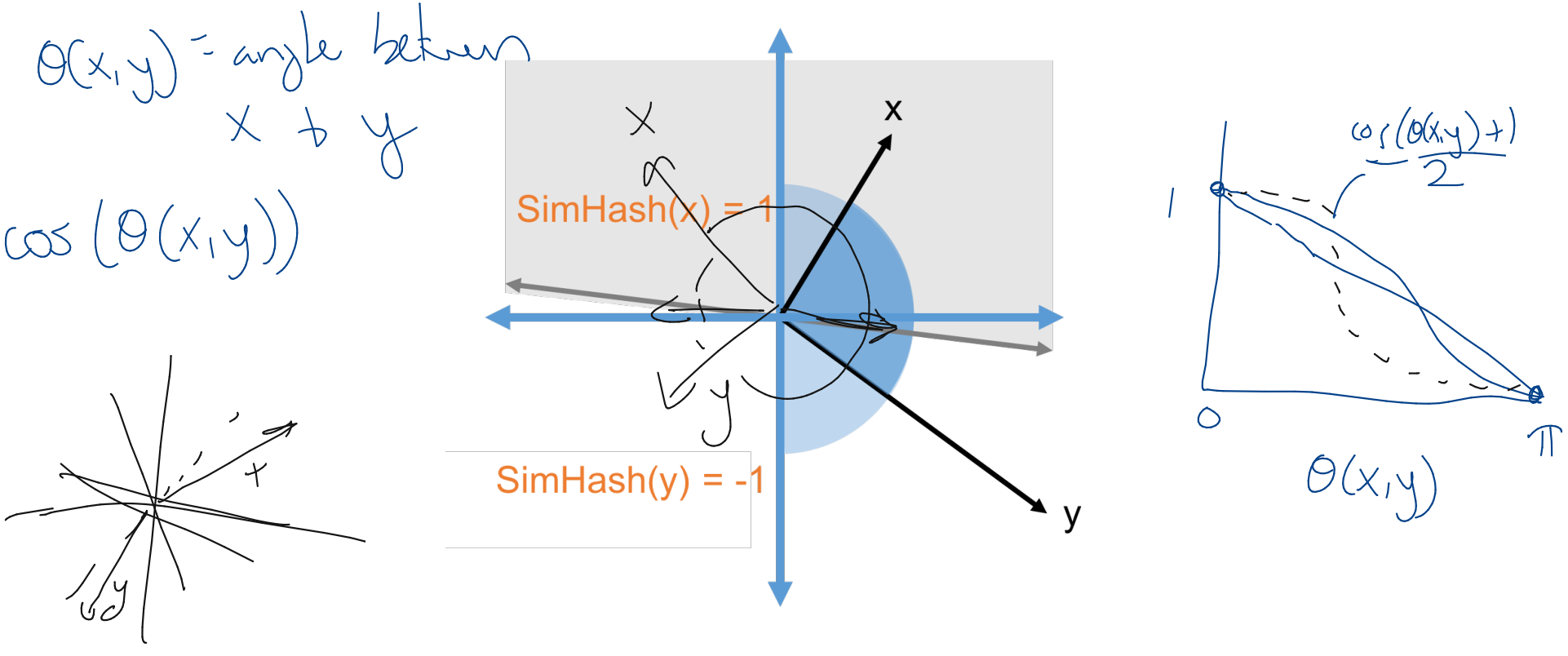


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- $\Pr[SimHash(x) \neq SimHash(y)] = \frac{\theta(x,y)}{\pi}$
- $\Pr[SimHash(x) = SimHash(y)] = 1 - \frac{\theta(x,y)}{\pi} \approx \frac{\cos(\theta(x,y))+1}{2}$

Questions on MinHash and Locality Sensitive Hashing?