COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2024.

Lecture 6

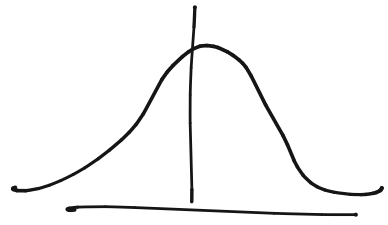
Logistics

- Problem Set 1 is due tomorrow at 11:59pm in Gradescope.
- Remember to make one submission per group, but all group members must be enrolled in Gradescope and have their names added to the submission.
- Submit 4 of the 5 questions and just submit nothing for the challenge problem that you didn't complete.
- Quiz 3 is due Monday at 8pm.

Last Time

Last Class:

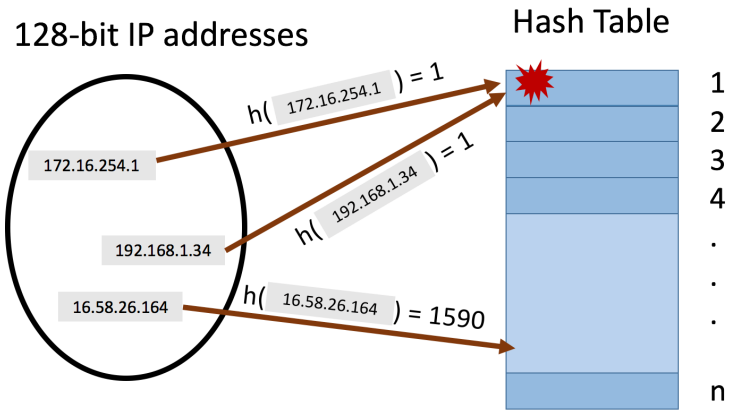
- Higher moment bounds and exponential concentration bounds
- Bernstein inequality and the Chernoff bound
- Connection to the central limit theorem.



This Class:

- Finish example application of exponential concentration bounds.
- Bloom filters: random hashing to maintain a large set in small space.

Application to Random Hashing



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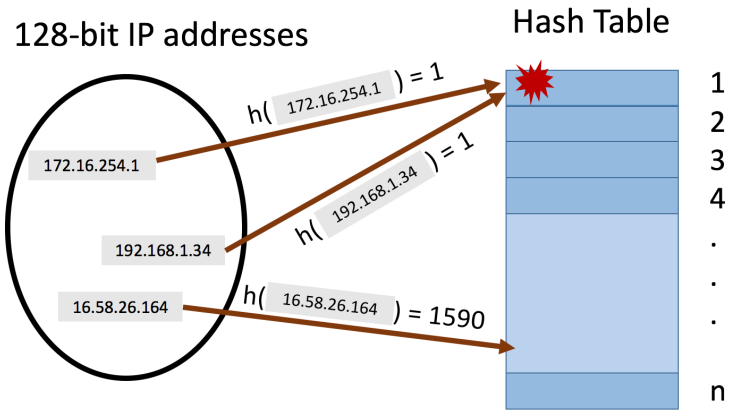

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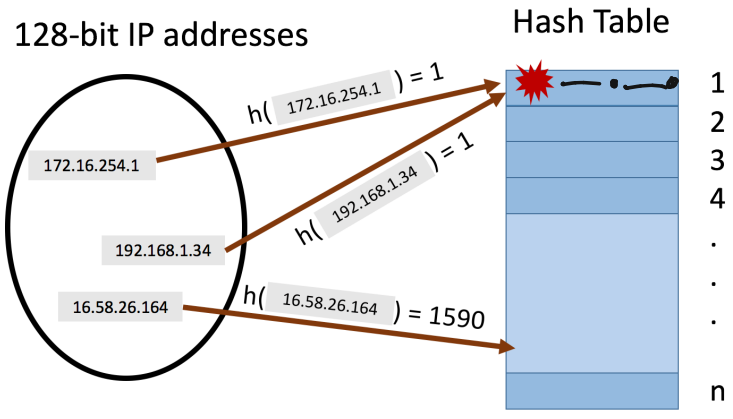

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What will be the maximum number of items hashed into the same location?

Maximum Load in Randomized Hashing

Let $\mathbf{S}_i$ be the number of items hashed into position i and $\mathbf{S}_{i,j}$ be 1 if x_j is hashed into bucket i ($\mathbf{h}(x_j) = i$) and 0 otherwise.

m: total number of items hashed and size of hash table. $x_1, \ldots, x_m$: the items.
$\mathbf{h}$: random hash function mapping $x_1, \ldots, x_m \to [m]$.

Maximum Load in Randomized Hashing

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$$\mathbb{E}[\mathsf{S}_i] = \sum_{j=1}^{m} \mathbb{E}[\mathsf{S}_{i,j}] = m \cdot \frac{1}{m} = 1$$

$\Pr(S_i \geq c) \leq ?$

↳ Markov

↳ Chebyshev / Bernstein

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By the Chernoff Bound: for any $\delta \geq 0$,

$$\Pr(\mathsf{S}_i \geq 1 + \delta) \leq \Pr\left(\left|\sum_{i=1}^{n} \mathsf{S}_{i,j} - 1\right| \geq \delta \cdot \mu\right) \leq 2\exp\left(-\frac{\delta^2}{2+\delta}\right)$$

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Set $\delta = 5 \log m$. Gives:

$$\Pr(S_i \geq (5 \log m) + 1) \leq 2\exp\left(-\frac{(5 \log m)^2}{2 + 5 \log m}\right).$$

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Set $\delta = 5\log m$. Gives:

$$\Pr(S_i \geq 5\log m + 1) \leq 2\exp\left(-\frac{(5\log m)^2}{2+5\log m}\right) \leq 2\exp(-3\log m) \leq \frac{2}{m^3}.$$

$$\frac{25(\log m)^2}{2+5\log m} \geq \frac{25(\log m)^2}{7\log m} \geq 3\log m$$

$$\leq 2\log(n)$$

using $m \geq n$

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Apply Union Bound:

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$$\leq \sum_{i=1}^{m} \Pr(S_i \geq 5\log m + 1) \leq m \cdot \frac{2}{m^3} = \frac{2}{m^2}.$$

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- Using Chebyshev's inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability (good exercise).

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- So, even with a simple linked list to store the items in each bucket, worst case query time is $O(\log m)$.
- Using Chebyshev's inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability (good exercise).
- The Chebyshev bound holds even with a pairwise independent hash function. The stronger Chernoff-based bound can be shown to hold with a *k-wise independent hash function* for $k = O(\log m)$.

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$\delta = 0.05$

- Allow small probability $\delta > 0$ of false positives. I.e., for any x,

$$\Pr(\textit{query}(x) = 1 \text{ and } x \notin S) \leq \delta.$$

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Solution: Bloom filters (repeated random hashing). Will use much less space than a hash table.

Bloom Filters

Chose k independent random hash functions $\mathbf{h}_1, \ldots, \mathbf{h}_k$ mapping the universe of elements $U \to [m]$.

- Maintain an array A containing m bits, all initially 0.
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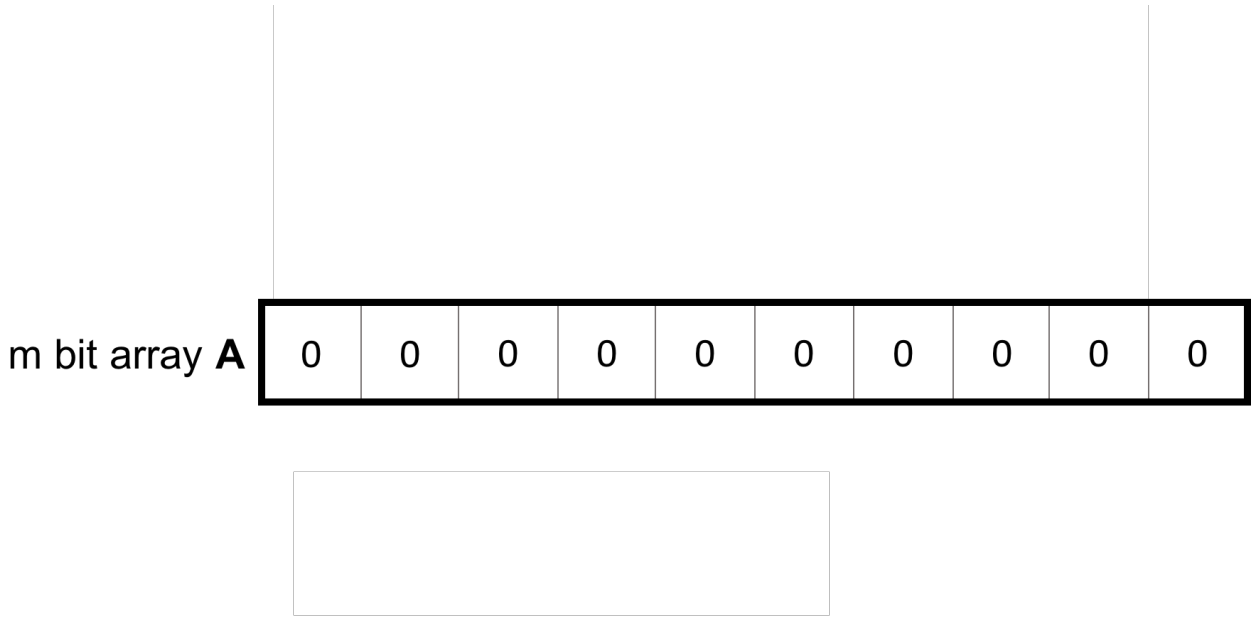

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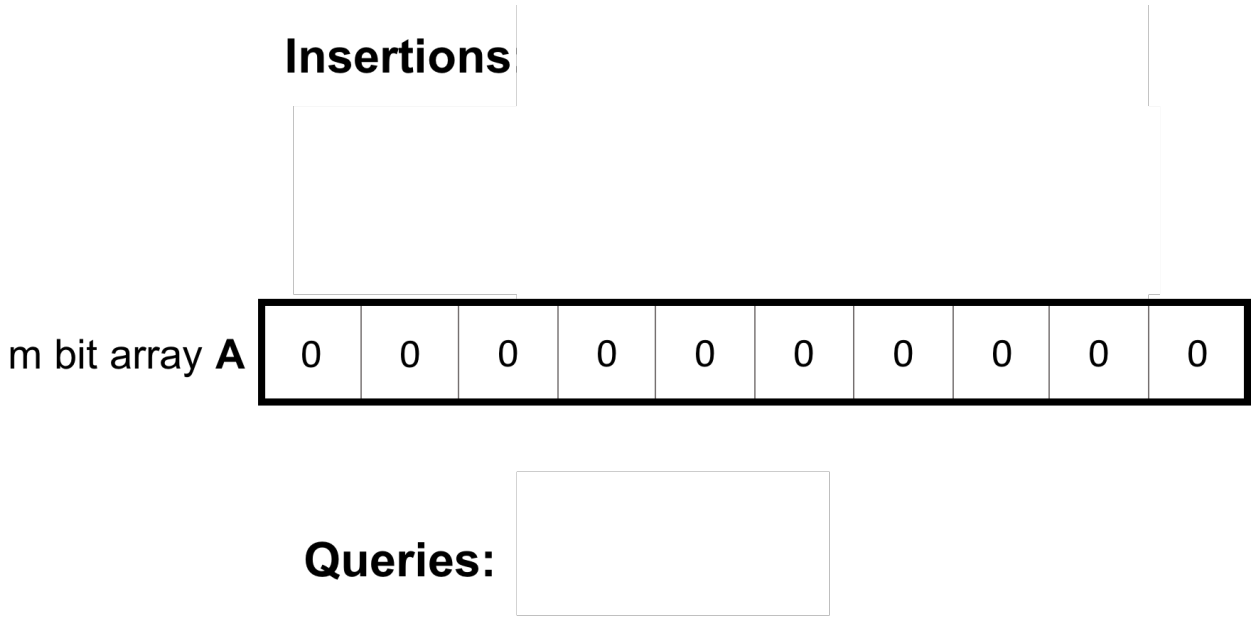

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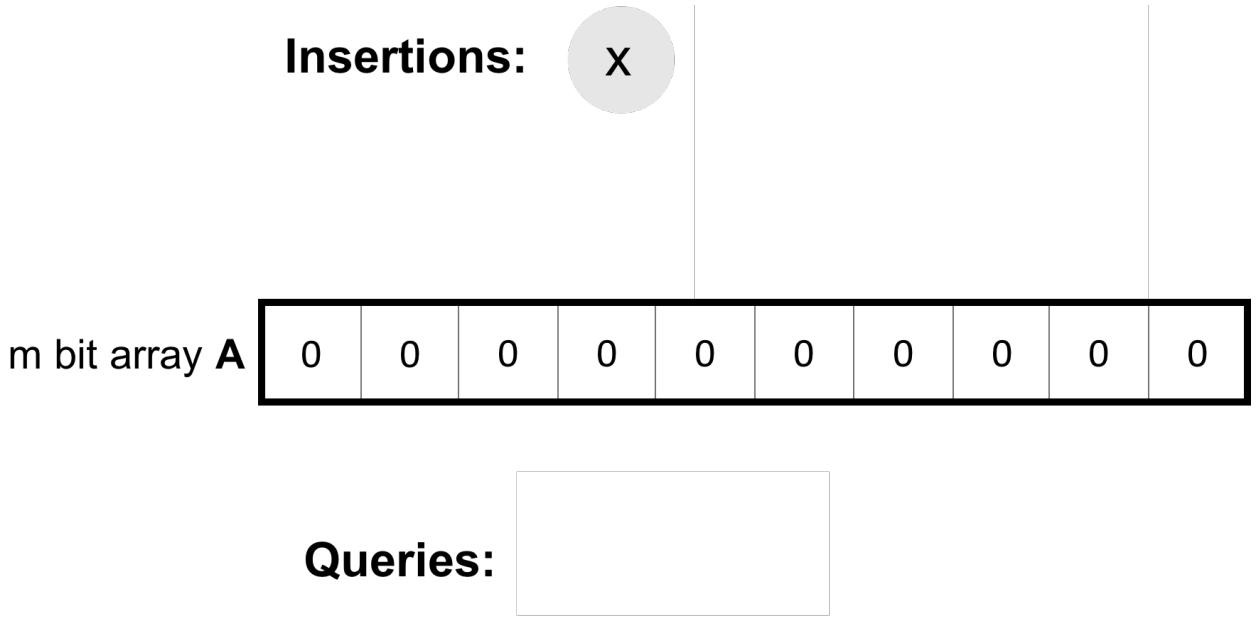

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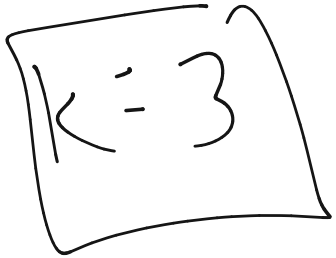

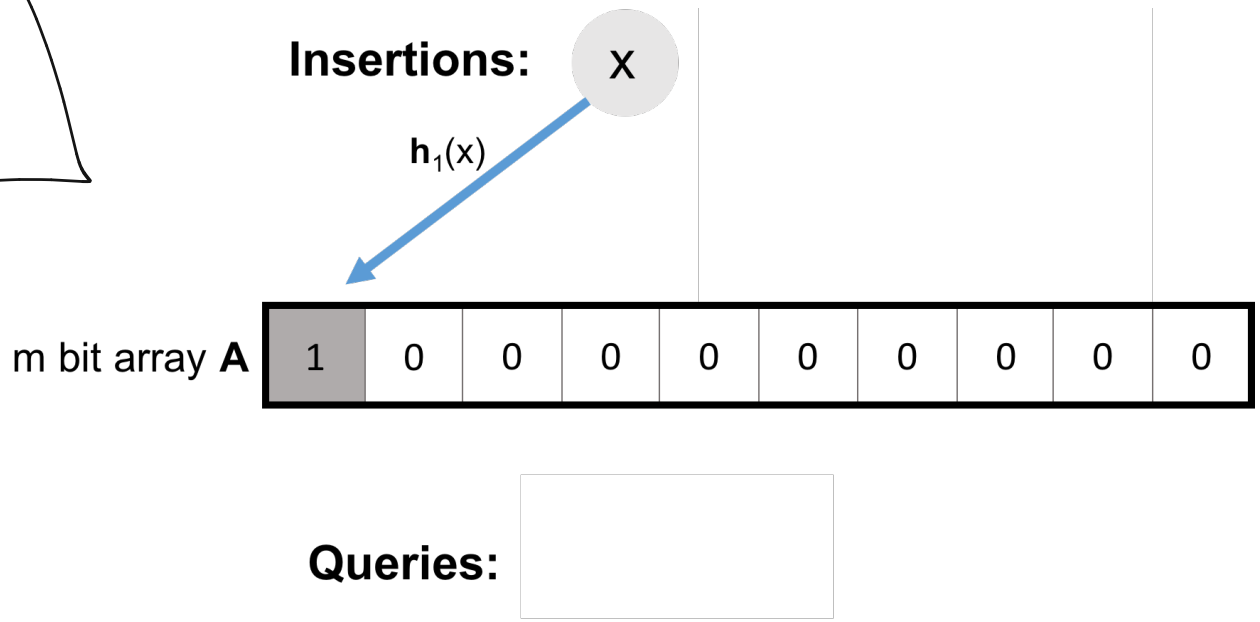

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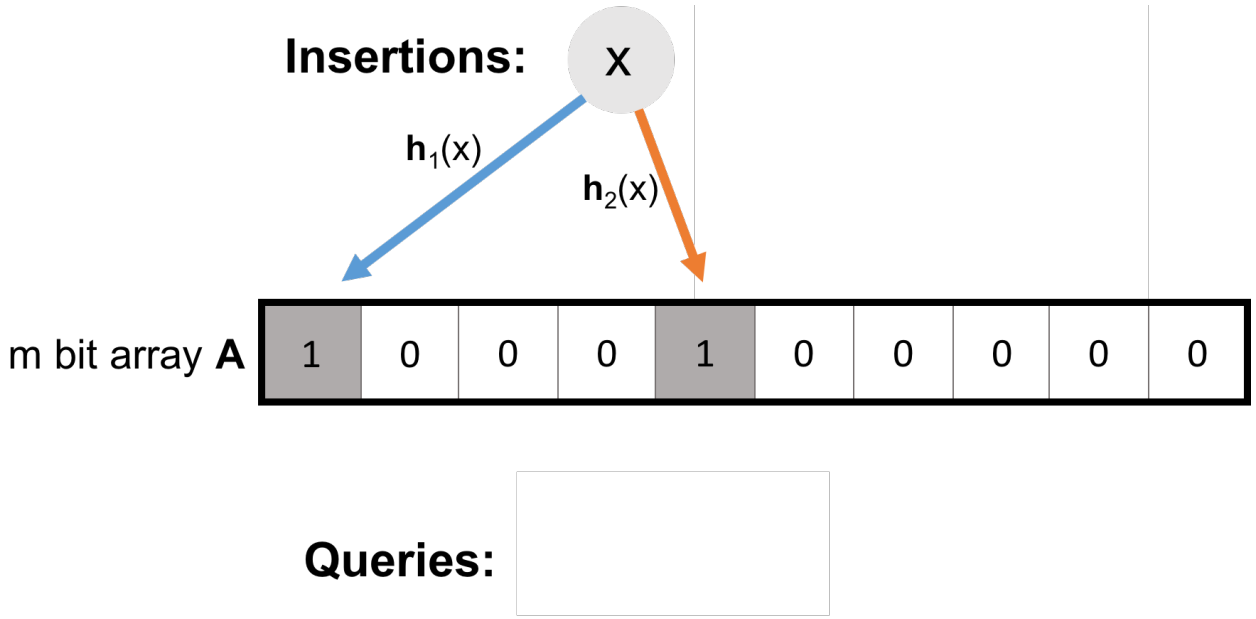

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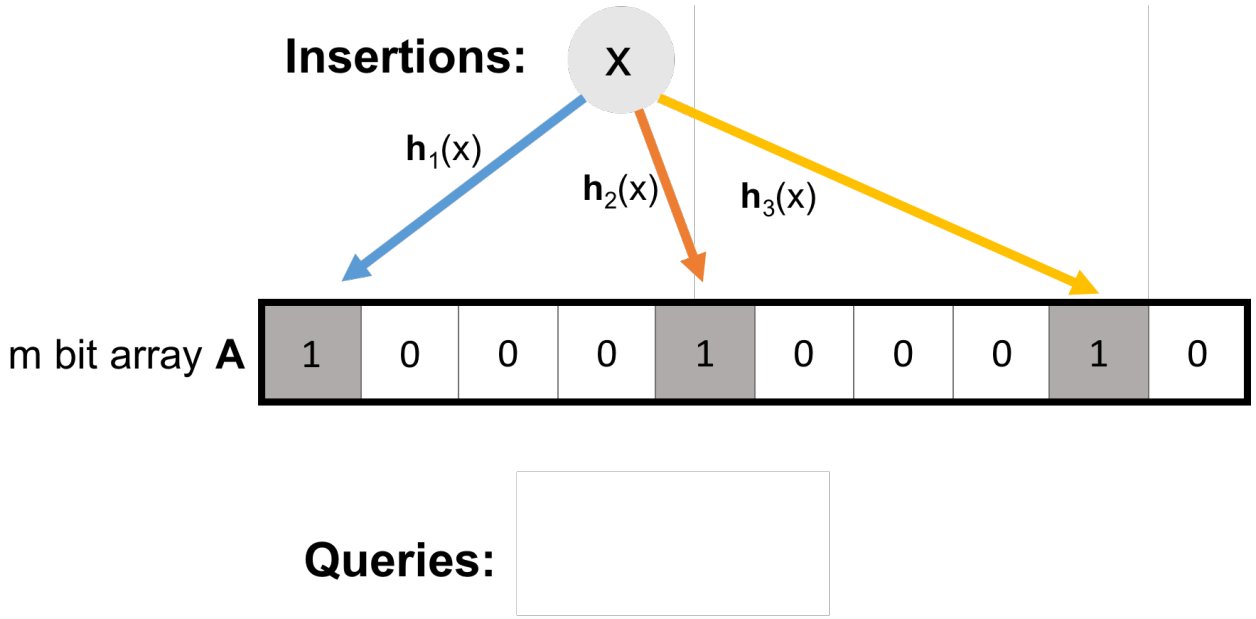

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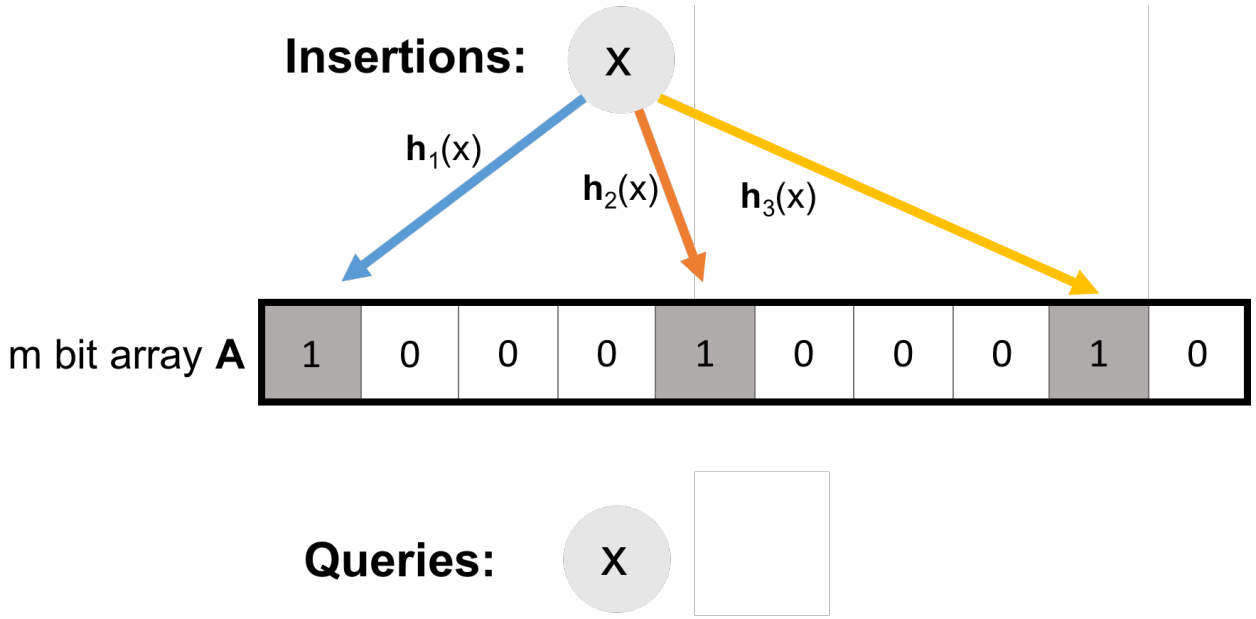

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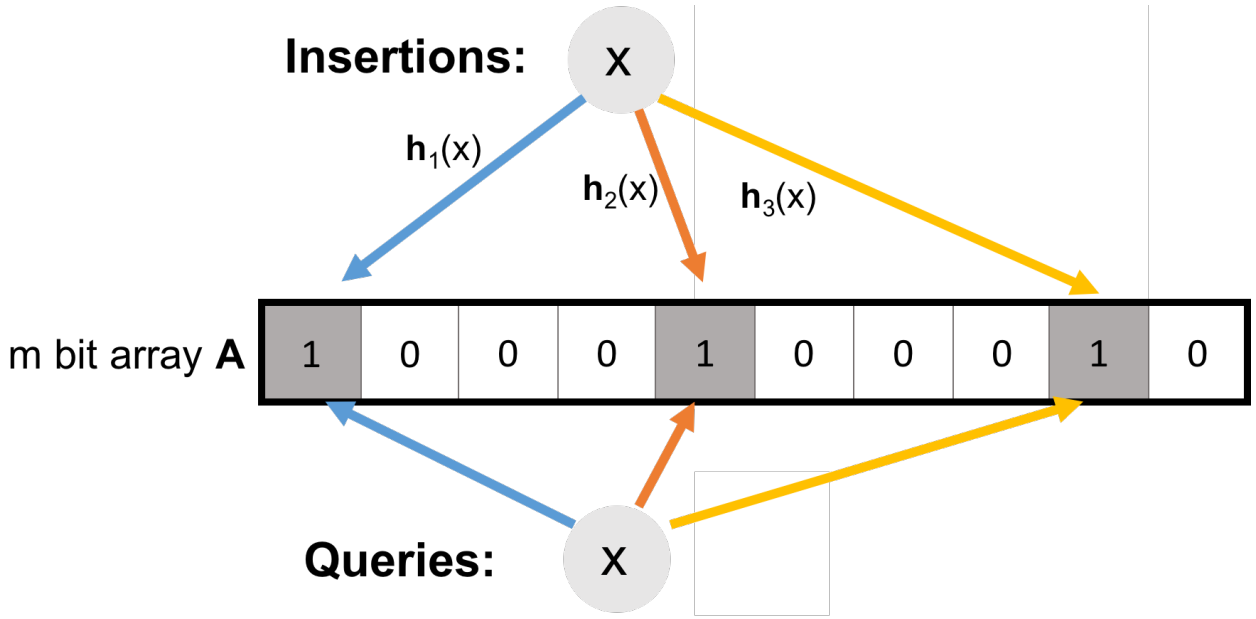

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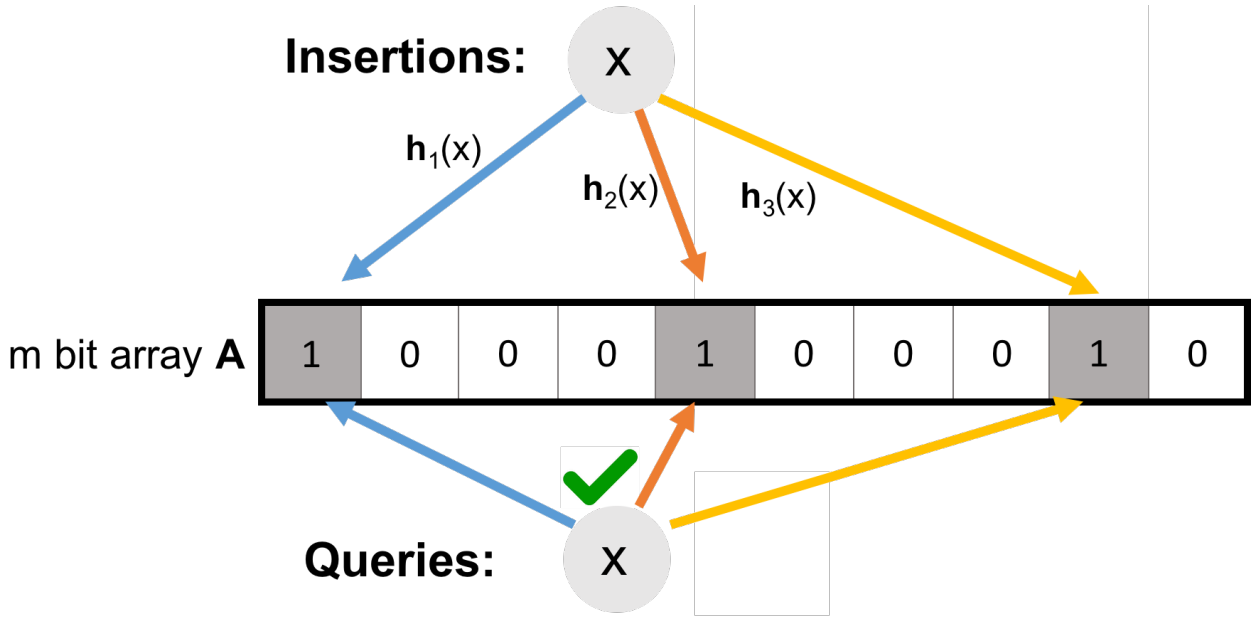

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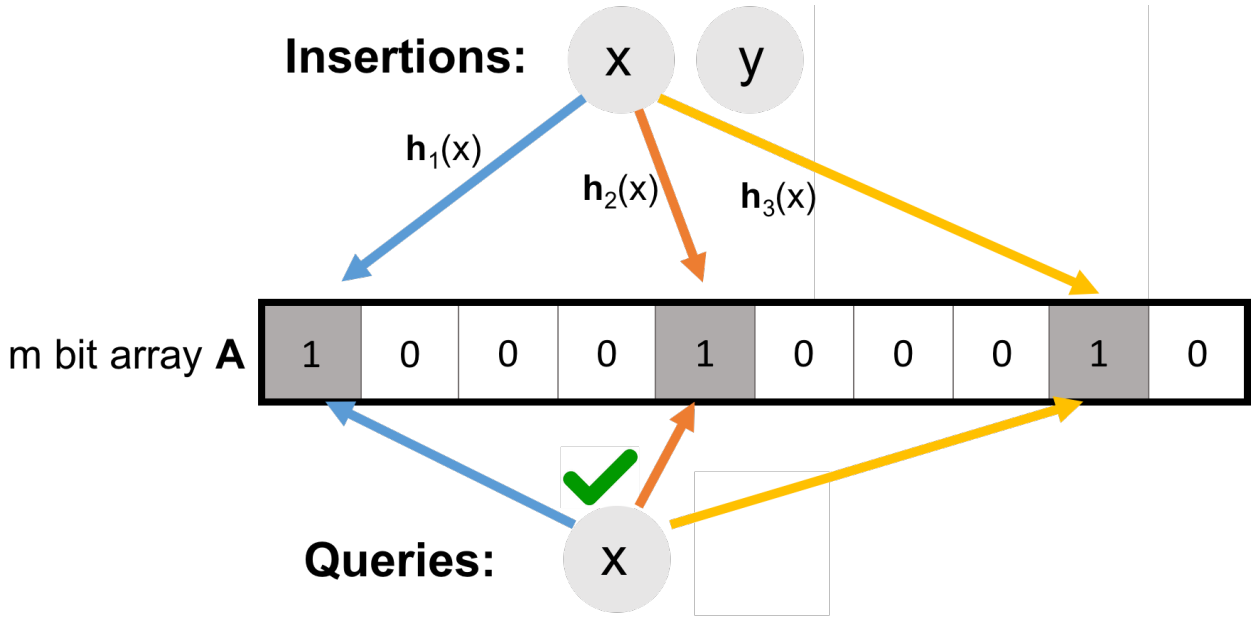

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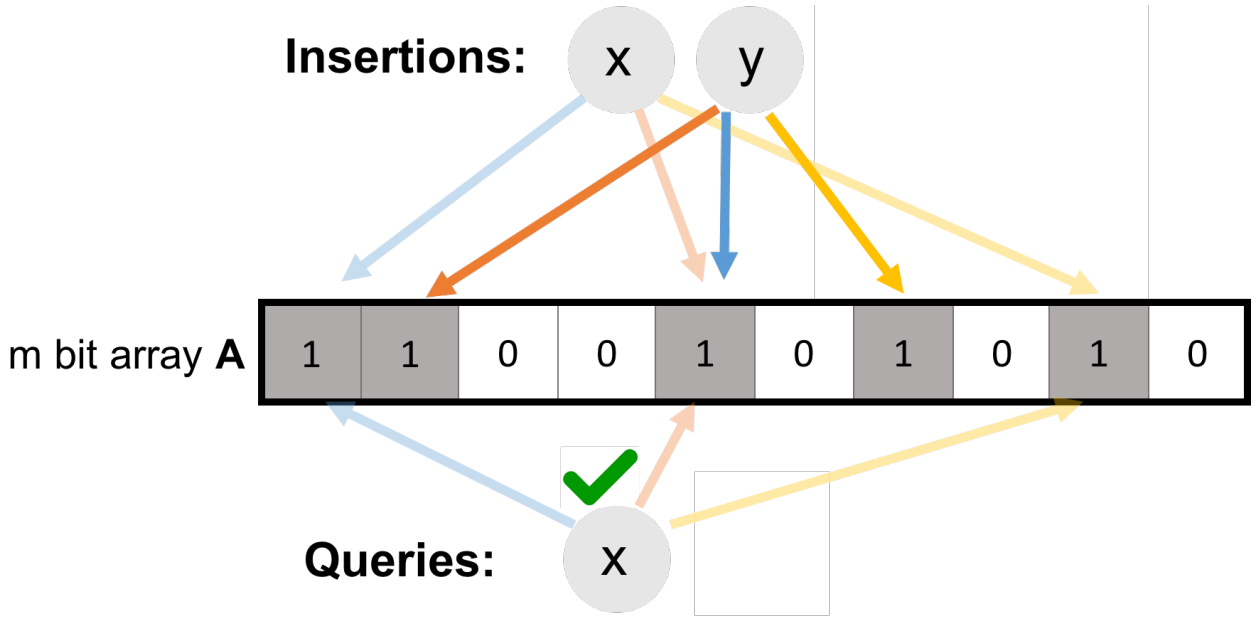

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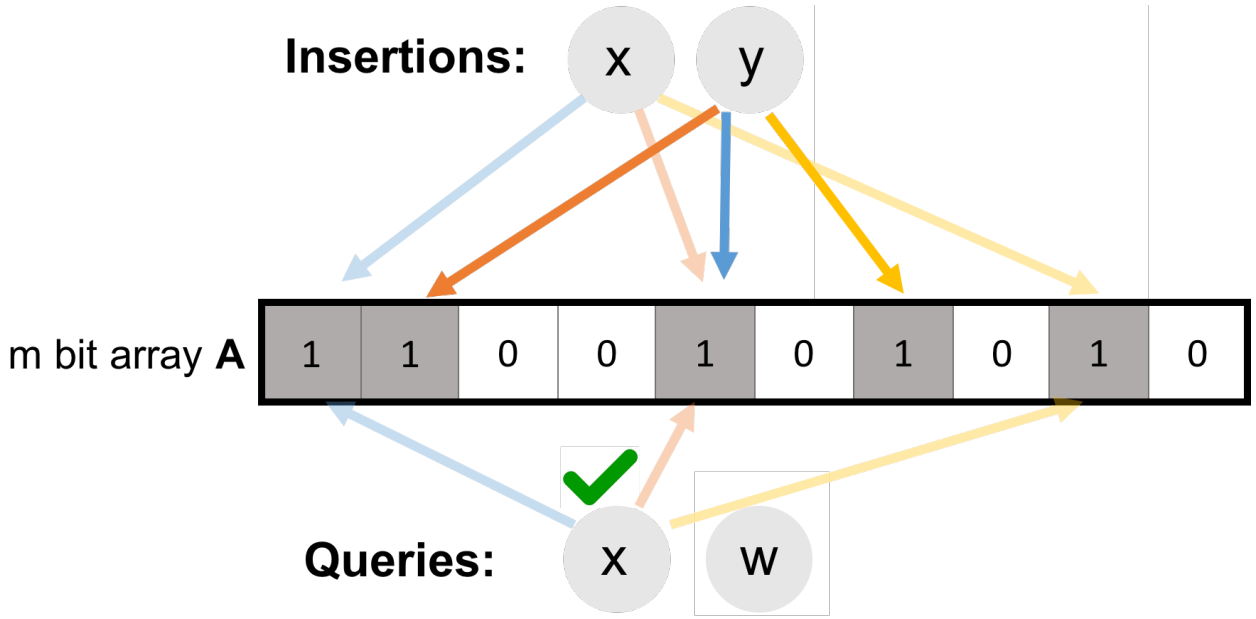

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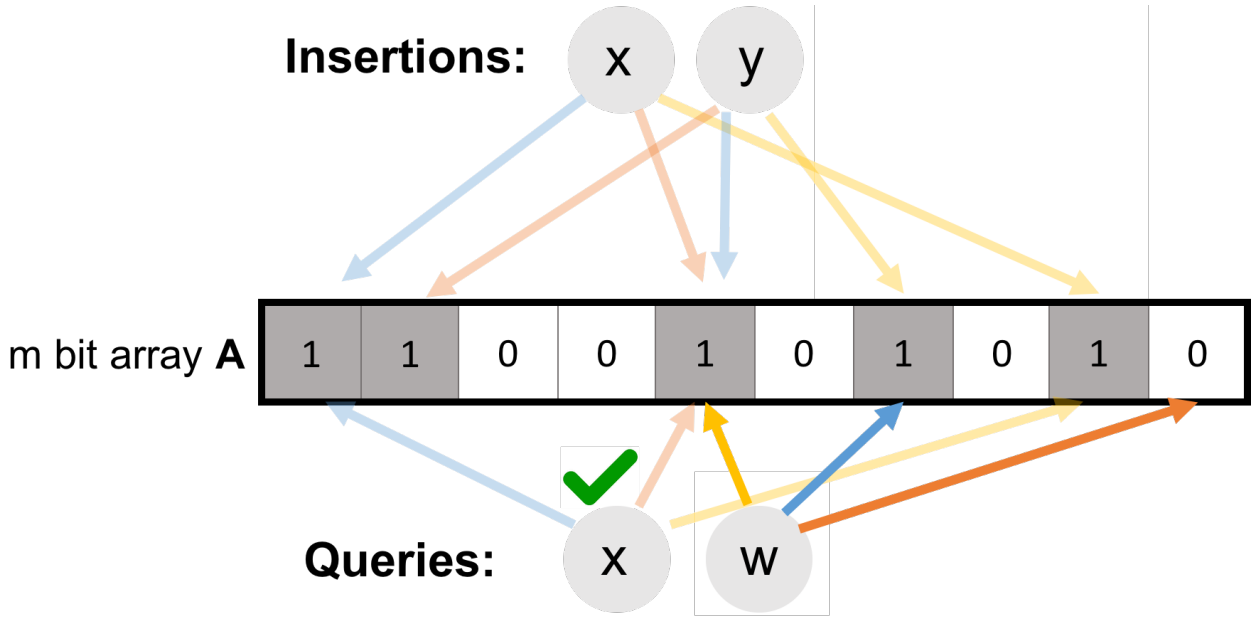

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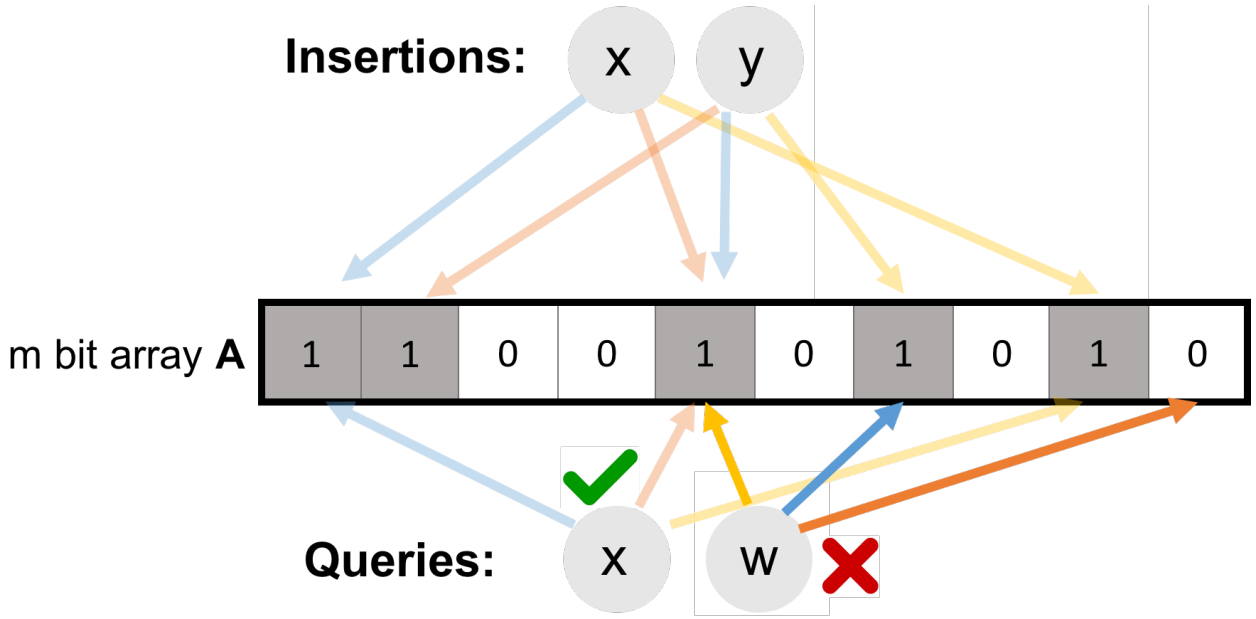

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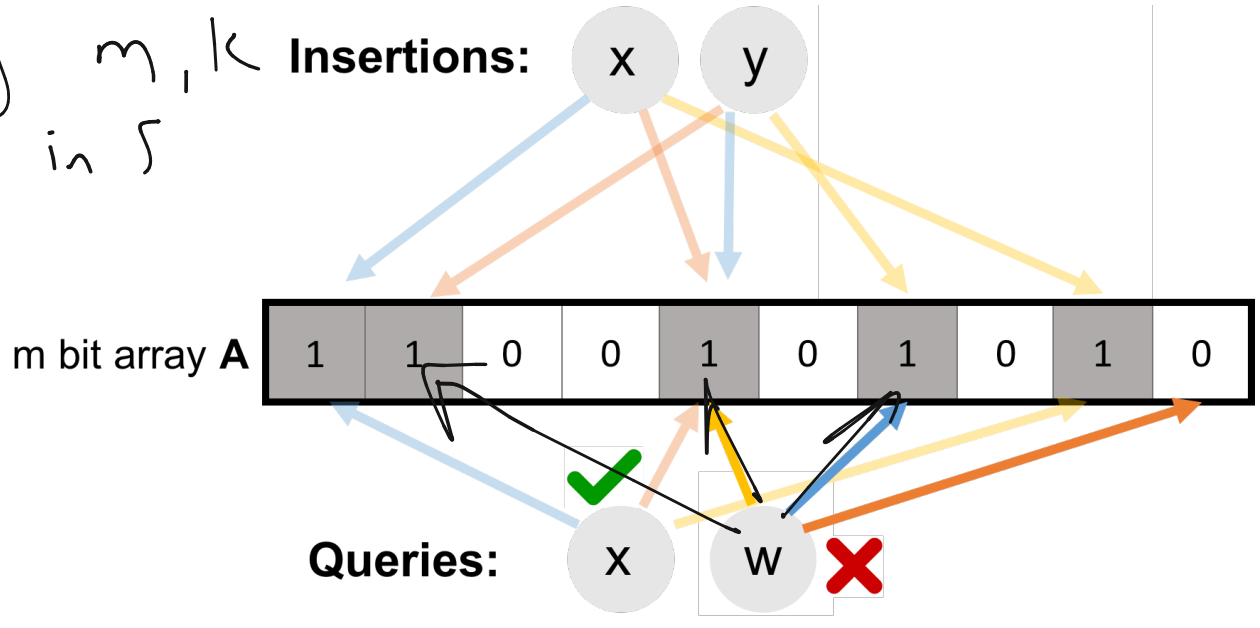
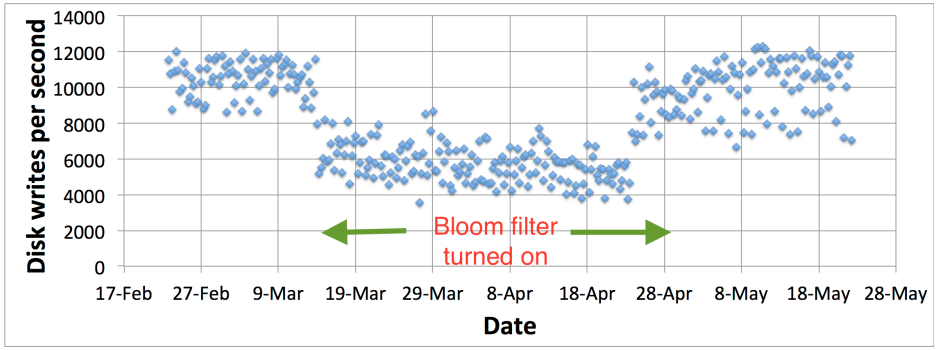
Size array m, k
$\Rightarrow$ #bits in S and d



No false negatives. False positives more likely with more insertions.

Applications: Caching

Akamai (Boston-based company serving 15 – 30% of all web traffic) applies bloom filters to prevent caching of 'one-hit-wonders' – pages only visited once fill over 75% of cache.



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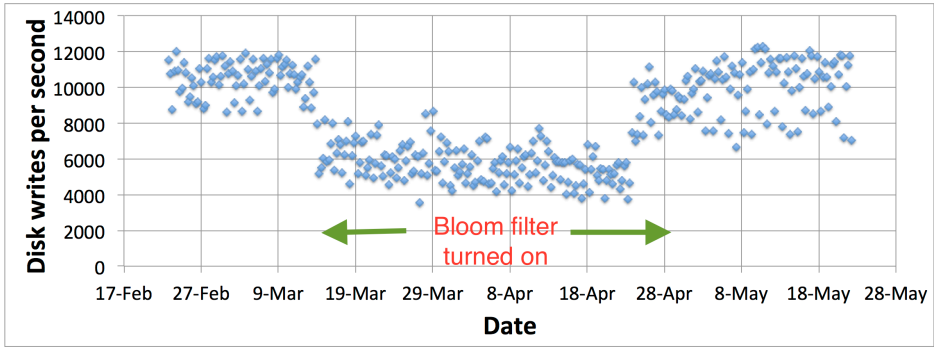

- When url x comes in, if $query(x) = 1$, cache the page at x. If not, run $insert(x)$ so that if it comes in again, it will be cached.

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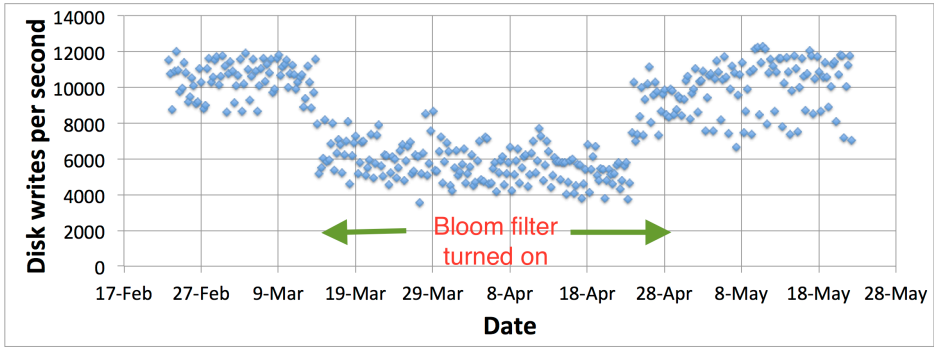

- When url x comes in, if $query(x) = 1$, cache the page at x. If not, run $insert(x)$ so that if it comes in again, it will be cached.
- **False positive:** A new url (possible one-hit-wonder) is cached. If the bloom filter has a false positive rate of $\delta = .05$, the number of cached one-hit-wonders will be reduced by at least 95%.

Applications: Databases

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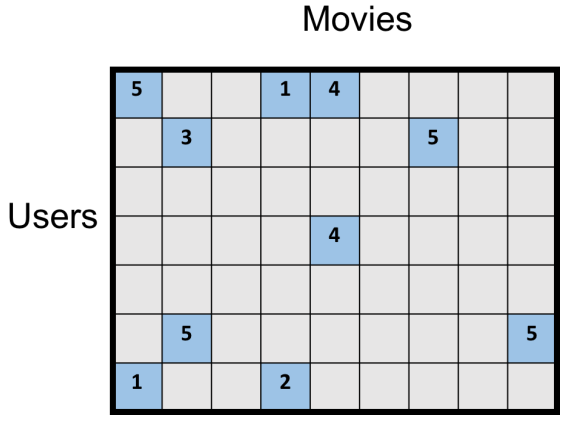

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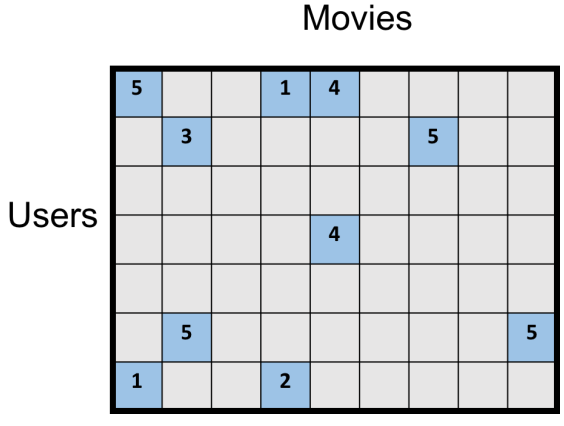

- When a new rating is inserted for $(user_x, movie_y)$, add $(user_x, movie_y)$ to a bloom filter.
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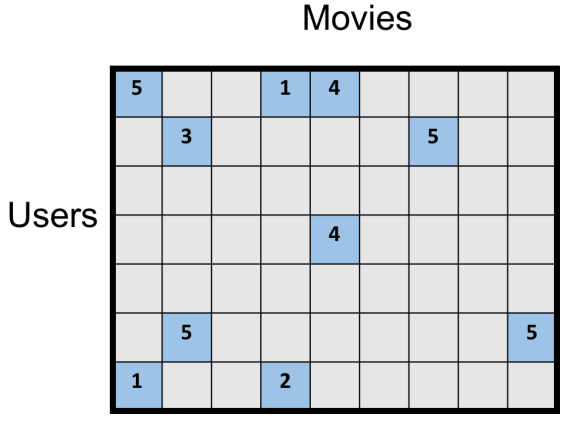

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- Before reading $(user_x, movie_y)$ (possibly via an out of memory access), check the bloom filter, which is stored in memory.
- **False positive:** A read is made to a possibly empty cell. A $\delta = .05$ false positive rate gives a 95% reduction in these empty reads.

More Applications

- **Database Joins:** Quickly eliminate most keys in one column that don't correspond to keys in another.
- **Recommendation systems:** Bloom filters are used to prevent showing users the same recommendations twice.
- **Spam/Fraud Detection**:
  - Bit.ly and Google Chrome use bloom filters to quickly check if a url maps to a flagged site and prevent a user from following it.
  - Can be used to detect repeat clicks on the same ad from a single IP-address, which may be the result of fraud.
- **Digital Currency:** Some Bitcoin clients use bloom filters to quickly pare down the full transaction log to transactions involving bitcoin addresses that are relevant to them (SPV: simplified payment verification).

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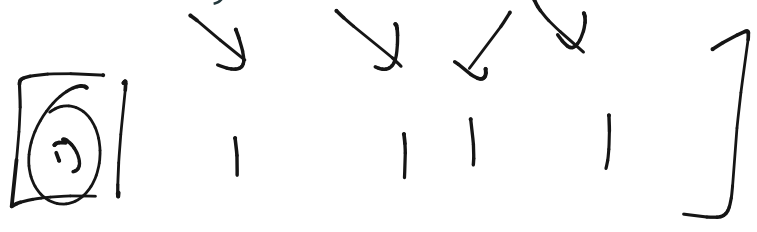

insert n elements, each mark k random positions as 1

$\frac{n \cdot k}{m}$ $\left(1 - \frac{k}{m}\right)^n$

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$$\Pr(A[i] = 0) = \Pr\left(\mathbf{h}_1(x_1) \neq i \cap \ldots \cap \mathbf{h}_k(x_1) \neq i \cap \mathbf{h}_1(x_2) \neq i \ldots \cap \mathbf{h}_k(x_2) \neq i \cap \ldots\right)$$

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$$\left(1 - \frac{1}{m}\right)^{m} \approx \frac{1}{e}$$

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Actually Incorrect! Dependent events.

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Correct Analysis Sketch

Step 1: To avoid dependence issues, condition on the event that the A has t zeros in it after n insertions, for some $t \leq m$. For a non-inserted element w, after conditioning on this event we correctly have:

$$\begin{aligned}\Pr(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1) \\ = \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1).\end{aligned}$$

I.e., the events $A[\mathbf{h}_1(w)] = 1$,..., $A[\mathbf{h}_k(w)] = 1$ are independent conditioned on the number of bits set in A. Why?

- Conditioned on this event, for any j, since $\mathbf{h}_j$ is a fully random hash function, $\Pr(A[\mathbf{h}_j(w)] = 1) = 1 - \frac{t}{m}$.
- Thus conditioned on this event, the false positive rate is $\left(1 - \frac{t}{m}\right)^k$.
- It remains to show that $\frac{t}{m} \approx e^{-\frac{kn}{m}}$ with high probability. We already have that $\mathbb{E}[\frac{t}{m}] = \frac{1}{m}\sum_{i=1}^{m} \Pr(A[i] = 0) \approx e^{-\frac{kn}{m}}$.

Correct Analysis Sketch

Need to show that the number of zeros t in A after n insertions is bounded by $O\left(e^{-\frac{kn}{m}}\right)$ with high probability.

Can apply Theorem 2 of:

`http://cglab.ca/~morin/publications/ds/bloom-submitted.pdf`