COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2024.

Lecture 2

Reminder

Reminders:

- Remember to sign up for Piazza.
- Find homework teammates (see Piazza Post) and sign up for Gradescope (code on course website).
- Week 1 Quiz will be available after class and is due **Monday at 8:00pm**.
- Let me know if you see any issues with the quiz. This is my first time giving a quiz over Canvas.

Overview

Last Class:

- Basic probability review. See course site for links to resources to refresh your probability background.
- Start on linearity of expectation and variance.

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Today:

- Proofs for linearity of expectation and variance.
- Algorithmic applications.
- Introduce Markov's inequality a fundamental concentration bound that let us prove that a random variable lies close to its expectation with good probability.
- Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.

Linearity of Expectation

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Proof:

s = 1 , t = 6

$$\mathbb{E}[X + Y] = \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot (s + t)$$

{1,..6} , {1,..6}

$\frac{1}{36}$

7

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$$= \sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot s + \sum_{t \in T} \sum_{s \in S} \Pr(X = s \cap Y = t) \cdot t$$

Pr(X = s)   Pr(Y = t)

X Y
(1, 1) 1/36
+
(1, 2) 1/36
+
⋮
(1, 6) 1/36
= 1/6

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&= \sum_{s\in S}\sum_{t\in T} \Pr(X = s \cap Y = t)\cdot s + \sum_{t\in T}\sum_{s\in S} \Pr(X = s \cap Y = t)\cdot t \\
&= \underbrace{\sum_{s\in S} \Pr(X = s)\cdot s}_{\mathbb{E}[X]} + \underbrace{\sum_{t\in T} \Pr(Y = t)\cdot t}_{\mathbb{E}[Y]}
\end{aligned}
$$

(law of total probability)

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\end{aligned}$$

(law of total probability)

$$= \mathbb{E}[X] + \mathbb{E}[Y].$$

Linearity of Variance

$$\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$$

Linearity of Variance

$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ when X and Y are uncorrelated, and in particular, when they are independent.

independent $\Rightarrow$ uncorrelated
$\Rightarrow$ linearity of
variance

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Claim 1: (exercise) $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$ (via linearity of expectation)

$\text{Var}[X] = \mathbb{E}[(X - \mathbb{E}X)^2]$

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$$\sum_{s \in S} \sum_{t \in T} \Pr(X = s \cap Y = t) \cdot s \cdot t$$

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$$\text{Var}[X+Y] = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2$$

$$\mathbb{E}[X^2 + 2XY + Y^2] \qquad (\mathbb{E}[X] + \mathbb{E}[Y])^2$$

$$\mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}X\,\mathbb{E}Y - \mathbb{E}Y^2$$

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$X = Y$

$\mathrm{Var}(X+Y) = \mathrm{Var}(2X)$

$= 4\,\mathrm{Var}(X)$

$\neq 2[\mathrm{Var}(X) + \mathrm{Var}(Y)]$

$\mathrm{Var}(X) + \mathrm{Var}(Y)$

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&= \mathrm{Var}[X] + \mathrm{Var}[Y].
\end{aligned}
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An Algorithmic Application

You have contracted with a new company to provide CAPTCHAS for your website.

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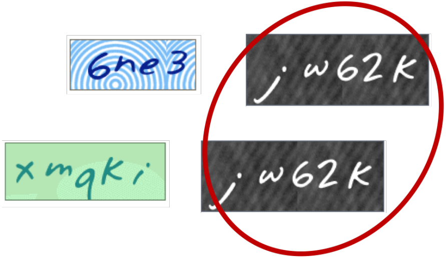
- They claim that they have a database of $1,000,000$ unique CAPTCHAS. A random one is chosen for each security check.
- You want to independently verify this claimed database size.
- You could make test checks until you see $1,000,000$ unique CAPTCHAS: would take $\geq 1,000,000$ checks!

An Algorithmic Application

An Idea: You run some test security checks and see if any duplicate CAPTCHAS show up. If you're seeing duplicates after not too many checks, the database size is probably not too big.



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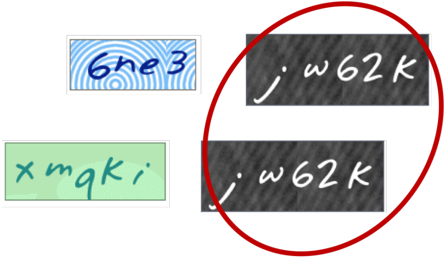

'Mark and recapture' method in ecology.

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D = 1

$\frac{\binom{m}{2}}{n} \quad \frac{m(m-1)}{2n}$

'Mark and recapture' method in ecology.

Think-Pair-Share: If you run m security checks, and there are n unique CAPTCHAS, how many pairwise duplicates do you see in expectation?

If e.g. the same CAPTCHA shows up three times, on your i^{th}, j^{th}, and k^{th} test, this is three duplicates: (i,j), (i,k) and (j,k).

Linearity of Expectation

Let $\mathbf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable.

n: number of CAPTCHAS in database, m: number of random CAPTCHAS drawn to check database size, $\mathbf{D}$: number of pairwise duplicates in m random CAPTCHAS

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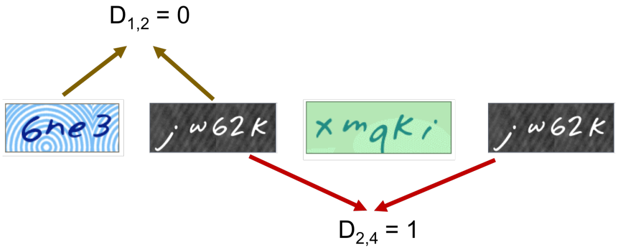

$m = 4$

$D_{4,2} = 1$

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Let $\mathsf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable. The number of pairwise duplicates (a random variable) is:

$$\mathsf{D} = \sum_{i,j \in [m], i \leq j} \mathsf{D}_{i,j}.$$

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$$\mathbb{E}[D] = \sum_{i,j \in [m], i<j} \mathbb{E}[D_{i,j}]. \sim \frac{1}{n}$$

$D_{ij} = 1$ w.p $\frac{1}{n}$

$D_{ij} = 0$ w.p $1 - \frac{1}{n}$

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$1, \ldots, n$

$$\mathbb{E}[\mathsf{D}] = \sum_{i,j \in [m], i<j} \mathbb{E}[\mathsf{D}_{i,j}].$$

For any pair $i, j \in [m], i < j$: $\mathbb{E}[\mathsf{D}_{i,j}] = \Pr[\mathsf{D}_{i,j} = 1] = \frac{1}{n}$.

$P(C_i = C_j)$

$P(C_i = C_j) = \sum_{k=1}^{n} Pr(C_i = C_j = k) = \frac{n}{n^2} = \frac{1}{n}$

$P(C_i = k \cap C_j = k)$

$P(C_i = k) \cdot P(C_j = k) = \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2}$

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$$\mathbb{E}[\mathsf{D}] = \sum_{i,j \in [m], i<j} \frac{1}{n} = \frac{\binom{m}{2}}{n} = \frac{m(m-1)}{2n}.$$

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$\approx \frac{m^2}{n}$

Note that the $\mathbf{D}_{i,j}$ random variables are not independent!

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Connection to the Birthday Paradox

If there are a 170 people in this room, each whose birthday we assume to be a uniformly random day of the 365 days in the year, how many pairwise duplicate birthdays do we expect there are?

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If there are a 170 people in this room, each whose birthday we assume to be a uniformly random day of the 365 days in the year, how many pairwise duplicate birthdays do we expect there are?

$$\mathbb{E}[D] = \frac{m(m-1)}{2n} = \frac{170 \cdot 169}{2 \cdot 365} \approx 39.$$

Linearity of Expectation

You take $m = 1000$ samples. If the database size is as claimed ($n = 1,000,000$) then expected number of duplicates is:

$$\mathbb{E}[D] = \frac{m(m-1)}{2n} = .4995$$

D_{ij} = 1 if $c_i = c_j$, 0 if $c_i \neq c_j$

D_{ij} are separate but correlated random variables

$D = \sum D_{ij}$

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You see 10 pairwise duplicates and suspect that something is up. But how confident can you be in your test?

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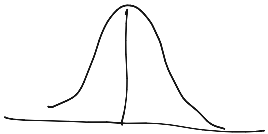
Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.



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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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The most fundamental concentration bound: **Markov's inequality.**

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For any non-negative random variable X and any $t > 0$:

$$\Pr[X \geq t] \leq \frac{\mathbb{E}[X]}{t}.$$

$t = s\,\mathbb{E}[X]$

$\Pr[X \geq s\,\mathbb{E}[X]] \leq \frac{1}{s}$

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I used nonnegativity

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$$\frac{\mathbb{E}[X]}{t} \geq \Pr(X \geq t)$$

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Useful form: $\Pr[X \geq t \cdot \mathbb{E}[X]] \leq \frac{1}{t}$.

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Useful form: $\Pr[X \geq t \cdot \mathbb{E}[X]] \leq \frac{1}{t}$.

The larger the deviation t, the smaller the probability.

Back to Our Application

Expected number of duplicate CAPTCHAS:

$\mathbb{E}[D] = \frac{m(m-1)}{2n} = .4995.$

You see $D = 10$ duplicates.

$$\Pr[D \geq 10] \leq \frac{\mathbb{E}[D]}{10} \leq \frac{.4995}{10} \leq \frac{1}{20}$$

n: number of CAPTCHAS in database ($n = 1{,}000{,}000$ claimed) , m: number of random CAPTCHAS drawn to check database size ($m = 1000$ in this example), D: number of pairwise duplicates in m random CAPTCHAS.

Back to Our Application

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$\mathbb{E}[\mathsf{D}] = \frac{m(m-1)}{2n} = .4995.$

You see $\mathsf{D} = 10$ duplicates.

Applying Markov's inequality, if the real database size is $n = 1,000,000$ the probability of this happening is:

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> n: number of CAPTCHAS in database ($n = 1,000,000$ claimed) , m: number of random CAPTCHAS drawn to check database size ($m = 1000$ in this example), D: number of pairwise duplicates in m random CAPTCHAS.

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Hash Tables

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Other solutions:

- binary tree $O(\log n)$
- tri
- linked list $O(n)$

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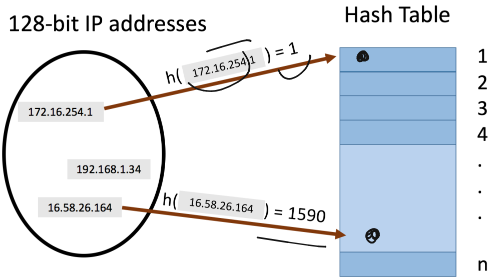
- *Static hashing* since we won't worry about insertion and deletion today.

Hash Tables



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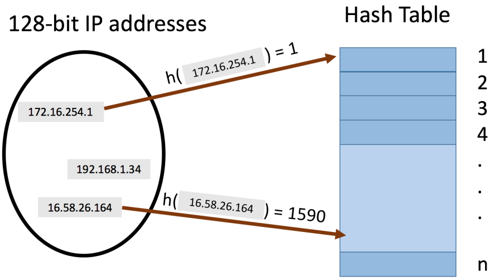

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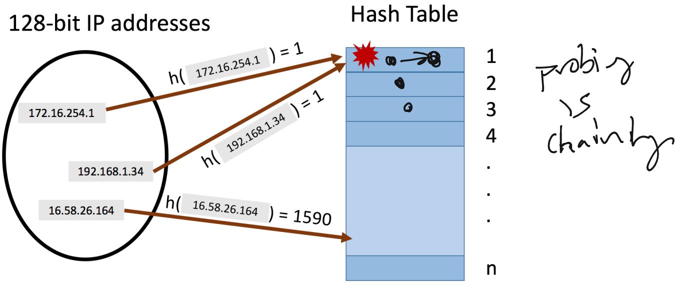

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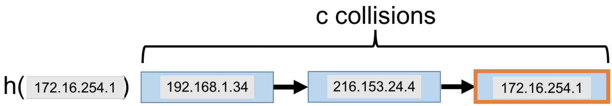
- Typically $|U| \gg n$. Many elements map to the same index.
- **Collisions:** when we insert m items into the hash table we may have to store multiple items in the same location (typically as a linked list).

Collisions

Query runtime: $O(c)$ when the maximum number of collisions in a table entry is c (i.e., must traverse a linked list of size c).



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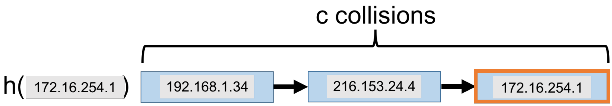

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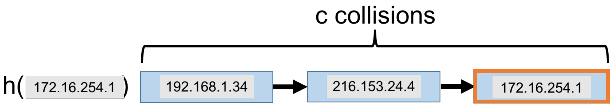

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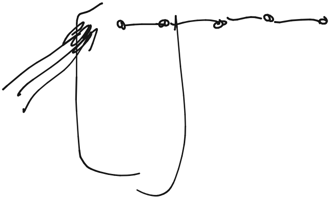

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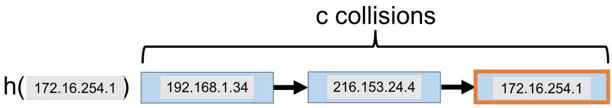

How Can We Bound c?

- In the worst case could have $c = m$ (all items hash to the same location).
- To avoid this, we'll assume the hash function is random, and so this event is very unlikely.

Random Hash Function

Let $\mathbf{h} : U \to [n]$ be a fully random hash function.

- I.e., for $x \in U$, $\Pr(\mathbf{h}(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ are independent for any two items $x \neq y$.

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- **Caveat 1:** It is *very expensive* to represent and compute such a random function. We will later see how a hash function computable in $O(1)$ time function can be used instead.
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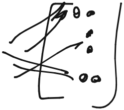

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Think-Pair-Share: Assuming we insert m elements into a hash table of size n using a fully random hash function, what is the expected total number of pairwise collisions?

↳ $\frac{\binom{m}{2}}{n}$

Linearity of Expectation

Let $\mathsf{C}_{i,j} = 1$ if items i and j collide $(\mathsf{h}(x_i) = \mathsf{h}(x_j))$, and 0 otherwise. The number of pairwise duplicates is:

$$\mathsf{C} = \sum_{i,j \in [m], i<j} \mathsf{C}_{i,j}.$$

x_i, x_j: pair of stored items, m: total number of stored items, n: hash table size, C: total pairwise collisions in table, h: random hash function.

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Identical to the CAPTCHA analysis!

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Collision Free Hashing

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- Say we have a lot of space. In particular, let $n = 4m^2$. Then: $\mathbb{E}[C] = \frac{m(m-1)}{8m^2} \leq \frac{1}{8}$.

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