COMPSCI 514: Algorithms for Data Science

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Lecture 14

Logistics

- Midterms and Problem Set 2 are being graded now.
- Problem Set 3 will be released shortly, likely due 11/8.

Summary

Last Few Classes: The Johnson-Lindenstrauss Lemma

- Reduce n data points in any dimension d to $O\left(\frac{\log n}{\epsilon^2}\right)$ dimensions and preserve all pairwise distances up to $1 \pm \epsilon$.
- Compression is linear via multiplication with a random, data oblivious, matrix (linear compression)
- Proved via the distributional JL-Lemma which shows that if $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ is a random matrix, $\mathbf{\Pi}\vec{y}_2 \approx \|\vec{y}\|$ for any y with high probability.
- Proof of distributional JL via linearity of expectation, linearity of variance, stability of the Gaussian distribution, and an exponential concentration bound for Chi-Squared random variables.

Summary

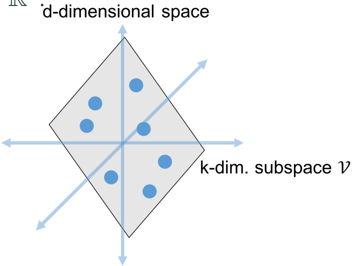
Next Few Classes: Low-rank approximation, the SVD, and principal component analysis (PCA).

- Reduce d-dimesional data points to a smaller dimension m.
- Like JL, compression is linear – by applying a matrix.
- Chose this matrix carefully, taking into account structure of the dataset.
- Can give better compression than random projection (although not directly comparable).

Will be using a fair amount of linear algebra: orthogonal basis, column/row span, eigenvectors, etc.

Embedding with Assumptions

Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie in any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.



Claim: Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns. For all $\vec{x}_i, \vec{x}_j$:

$$\|\mathbf{V}^T\vec{x}_i - \mathbf{V}^T\vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

- $\mathbf{V}^T \in \mathbb{R}^{k \times d}$ is a linear embedding of $\vec{x}_1, \ldots, \vec{x}_n$ into k dimensions with no distortion.

Dot Product Transformation

Claim: Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns. For all $\vec{x}_i, \vec{x}_j \in \mathcal{V}$:

$$\|\mathbf{V}^T \vec{x}_i - \mathbf{V}^T \vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

Dot Product Transformation

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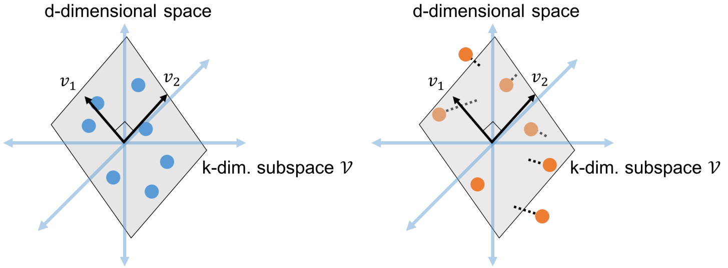
$$\|\mathbf{V}^T \vec{x}_i - \mathbf{V}^T \vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

Embedding with Assumptions

Main Focus of Upcoming Classes: Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie close to any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.



Letting $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns, $\mathbf{V}^T\vec{x}_i \in \mathbb{R}^k$ is still a good embedding for $x_i \in \mathbb{R}^d$. The key idea behind low-rank approximation and principal component analysis (PCA).

- How do we find $\mathcal{V}$ and $\mathbf{V}$?
- How good is the embedding?

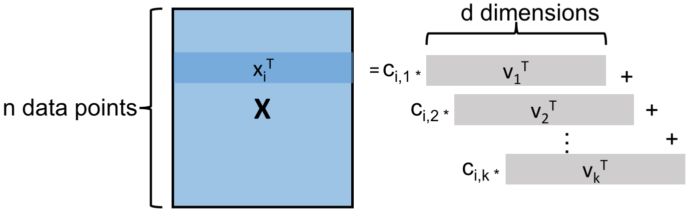
Low-Rank Factorization

Claim: $\vec{x}_1, \ldots, \vec{x}_n$ lie in a k-dimensional subspace $\mathcal{V} \Leftrightarrow$ the data matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ has rank $\leq k$.

- Letting $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$, can write $\vec{x}_i$ as:

$$\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + c_{i,2} \cdot \vec{v}_2 + \ldots + c_{i,k} \cdot \vec{v}_k.$$

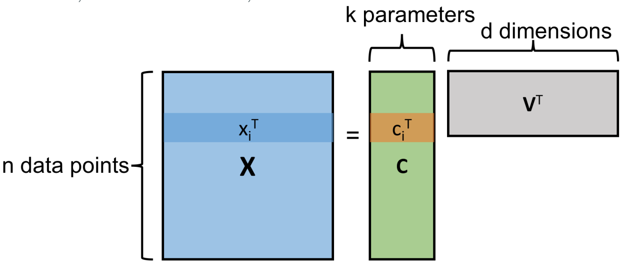
- So $\vec{v}_1, \ldots, \vec{v}_k$ span the rows of $\mathbf{X}$ and thus $\mathbf{rank}(\mathbf{X}) \leq k$.



$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Claim: $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ lie in a k-dimensional subspace $\mathcal{V}$ $\Leftrightarrow$ the data matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ has rank $\leq k$.

- Every data point $\vec{x}_i$ (row of $\mathbf{X}$) can be written as $\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + \ldots + c_{i,k} \cdot \vec{v}_k$.

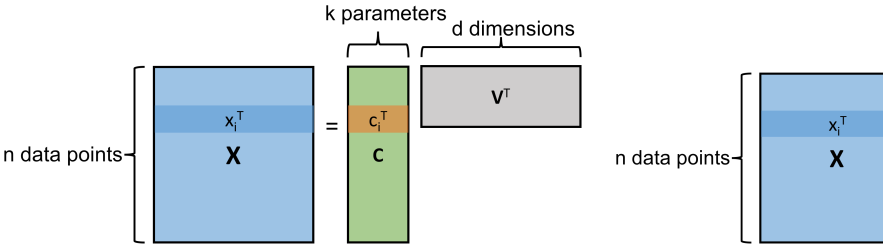


- $\mathbf{X}$ can be represented by $(n + d) \cdot k$ parameters vs. $n \cdot d$.
- The rows of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{V}$ $\implies$ the columns of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{C}$.

$\vec{x}_1, \ldots, \vec{x}_n$: data points (in $\mathbb{R}^d$), $\mathcal{V}$: k-dimensional subspace of $\mathbb{R}^d$, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Low-Rank Factorization

Claim: If $\vec{x}_1, \ldots, \vec{x}_n$ lie in a k-dimensional subspace with orthonormal basis $\mathbf{V} \in \mathbb{R}^{d \times k}$, the data matrix can be written as $\mathbf{X} = \mathbf{C}\mathbf{V}^T$.



Exercise: What is this coefficient matrix $\mathbf{C}$? **Hint:** Use that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

- $\mathbf{X} = \mathbf{C}\mathbf{V}^T \implies \mathbf{X}\mathbf{V} = \mathbf{C}\mathbf{V}^T\mathbf{V} \implies \mathbf{X}\mathbf{V} = \mathbf{C}$

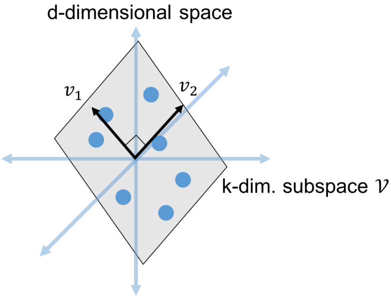
$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

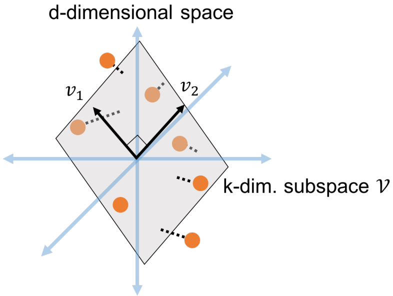
Projection View

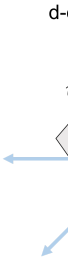
Claim: If $\vec{x}_1, \ldots, \vec{x}_n$ lie in a k-dimensional subspace $\mathcal{V}$ with orthonormal basis $\mathbf{V} \in \mathbb{R}^{d \times k}$, the data matrix can be written as

$$\mathbf{X} = \mathbf{C}\mathbf{V}^T \mathbf{X}\mathbf{V}\mathbf{V}^T.$$

- $\mathbf{V}\mathbf{V}^T$ is a projection matrix, which projects vectors onto the subspace $\mathcal{V}$.



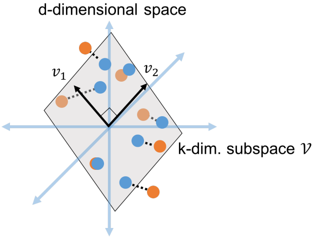




Low-Rank Approximation

Claim: If $\vec{x}_1, \ldots, \vec{x}_n$ lie close to a k-dimensional subspace $\mathcal{V}$ with orthonormal basis $\mathbf{V} \in \mathbb{R}^{d \times k}$, the data matrix can be approximated as:

$$\mathbf{X} \approx \mathbf{X}\mathbf{V}\mathbf{V}^T$$



Note: $\mathbf{X}\mathbf{V}\mathbf{V}^T$ has rank k. It is a low-rank approximation of $\mathbf{X}$.

$$\mathbf{X}\mathbf{V}\mathbf{V}^T = \underset{\mathbf{B} \text{ with rows in } \mathcal{V}}{\arg\min} \|\mathbf{X} - \mathbf{B}\|_F^2 = \sum_{i,j} (\mathbf{X}_{i,j} - \mathbf{B}_{i,j})^2.$$

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Low-Rank Approximation

So Far: If $\vec{x}_1, \ldots, \vec{x}_n$ lie close to a k-dimensional subspace $\mathcal{V}$ with orthonormal basis $\mathbf{V} \in \mathbb{R}^{d \times k}$, the data matrix can be approximated as:

$$\mathbf{X} \approx \mathbf{X}\mathbf{V}\mathbf{V}^T.$$

This is the closest approximation to $\mathbf{X}$ with rows in $\mathcal{V}$ (i.e., in the column span of $\mathbf{V}$).

- Letting $(\mathbf{X}\mathbf{V}\mathbf{V}^T)_i, (\mathbf{X}\mathbf{V}\mathbf{V}^T)_j$ be the i^{th} and j^{th} projected data points,
$$\|(\mathbf{X}\mathbf{V}\mathbf{V}^T)_i - (\mathbf{X}\mathbf{V}\mathbf{V}^T)_j\|_2 = \|[(\mathbf{X}\mathbf{V})_i - (\mathbf{X}\mathbf{V})_j]\mathbf{V}^T\|_2 = \|[(\mathbf{X}\mathbf{V})_i - (\mathbf{X}\mathbf{V})_j]\|_2.$$
- Can use $\mathbf{X}\mathbf{V} \in \mathbb{R}^{n \times k}$ as a compressed approximate data set.

Key question is how to find the subspace $\mathcal{V}$ and correspondingly $\mathbf{V}$.

$\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$: data points, $\mathbf{X} \in \mathbb{R}^{n \times d}$: data matrix, $\vec{v}_1, \ldots, \vec{v}_k \in \mathbb{R}^d$: orthogonal basis for subspace $\mathcal{V}$. $\mathbf{V} \in \mathbb{R}^{d \times k}$: matrix with columns $\vec{v}_1, \ldots, \vec{v}_k$.

Quick Exercise: Show that $\mathbf{V}\mathbf{V}^T$ is idempotent. I.e., $(\mathbf{V}\mathbf{V}^T)(\mathbf{V}\mathbf{V}^T)\vec{y} = (\mathbf{V}\mathbf{V}^T)\vec{y}$ for any $\vec{y} \in \mathbb{R}^d$.

Why does this make sense intuitively?

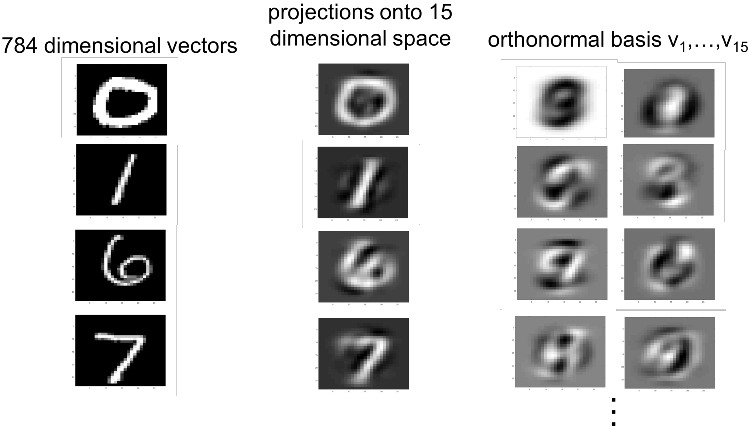
Less Quick Exercise: (Pythagorean Theorem) Show that:

$$\|\vec{y}\|_2^2 = \|(\mathbf{V}\mathbf{V}^T)\vec{y}\|_2^2 + \|\vec{y} - (\mathbf{V}\mathbf{V}^T)\vec{y}\|_2^2.$$

A Step Back: Why Low-Rank Approximation?

Question: Why might we expect $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ to lie close to a k-dimensional subspace?

- The rows of X can be approximately reconstructed from a basis of k vectors.



Question: Why might we expect $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ to lie close to a k-dimensional subspace?

- Equivalently, the columns of $\mathbf{X}$ are approx. spanned by k vectors.

Linearly Dependent Variables:

	bedrooms	bathrooms	sq.ft.	floors	list price	sale price
home 1	2	2	1800	2	200,000	195,000
home 2	4	2.5	2700	1	300,000	310,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
home n	5	3.5	3600	3	450,000	450,000

	bedrooms
home 1	2
home 2	4
⋮	⋮
home n	5