COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2024.

Lecture 14

Logistics

- Midterms and Problem Set 2 are being graded now.
- Problem Set 3 will be released shortly, likely due 11/8.
- Quiz due monday.

Summary

Last Few Classes: The Johnson-Lindenstrauss Lemma

- Reduce n data points in any dimension d to $O\left(\frac{\log n}{\epsilon^2}\right)$ dimensions and preserve all pairwise distances up to $1 \pm \epsilon$.
- Compression is linear via multiplication with a random, data oblivious, matrix (linear compression)
- Proved via the distributional JL-Lemma which shows that if $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ is a random matrix, $\mathbf{\Pi}\vec{y}_2 \approx \|\vec{y}\|$ for any y with high probability. $y = x_i - x_j$
- Proof of distributional JL via linearity of expectation, linearity of variance, stability of the Gaussian distribution, and an exponential concentration bound for Chi-Squared random variables.

Summary

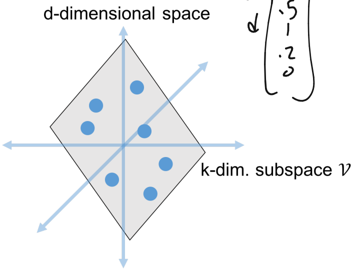
Next Few Classes: Low-rank approximation, the SVD, and principal component analysis (PCA).

- Reduce d-dimesional data points to a smaller dimension m.
- Like JL, compression is linear – by applying a matrix.
- Chose this matrix carefully, taking into account structure of the dataset.
- Can give better compression than random projection (although not directly comparable).

Will be using a fair amount of linear algebra: orthogonal basis, column/row span, eigenvectors, etc.

Embedding with Assumptions

Assume that data points $\vec{x}_1, \ldots, \vec{x}_n$ lie in any k-dimensional subspace $\mathcal{V}$ of $\mathbb{R}^d$.



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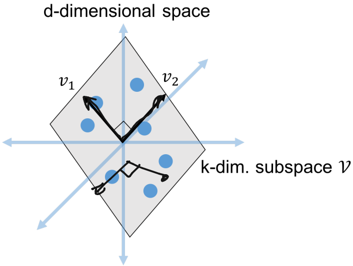

Claim: Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns. For all $\vec{x}_i, \vec{x}_j$:

$$\|\mathbf{V}^T\vec{x}_i - \mathbf{V}^T\vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

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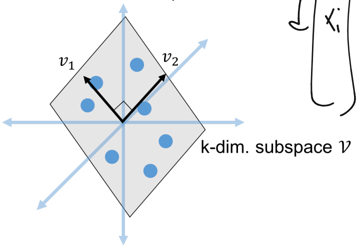

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$$\|\mathbf{V}^T\vec{x}_i - \mathbf{V}^T\vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

- $\mathbf{V}^T \in \mathbb{R}^{k \times d}$ is a linear embedding of $\vec{x}_1, \ldots, \vec{x}_n$ into k dimensions with no distortion.

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$$\|\mathbf{V}^T \vec{x}_i - \mathbf{V}^T \vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

for all i, $\exists\, c_i \in \mathbb{R}^k$

$$x_i = V c_i$$

$$x_i = \vec{v}_1 \cdot c_i(1) + \vec{v}_2\, c_i(2) \cdots + \vec{v}_k\, c_i(k)$$

$$\|V^T x_i - V^T x_j\|_2^2 = \|V^T V c_i - V^T V c_j\|_2^2 = \|c_i - c_j\|_2^2$$

$$(V^T V)_{ij} = \langle v_i, v_j \rangle = v_i^T v_j = 1 \text{ if } i = j,\quad 0 \text{ if } i \neq j$$

$$\|y\|_2 = \sqrt{\textstyle\sum y(i)^2}$$

$$\|y\|_2^2 = \textstyle\sum y(i)^2 = \langle y, y \rangle = y^T y$$

$$\|V c_i - V c_j\|_2^2 = \langle V(c_i - c_j), V(c_i - c_j) \rangle = (c_i - c_j)^T V^T V (c_i - c_j) = (c_i - c_j)^T (c_i - c_j) = \|c_i - c_j\|_2^2$$

Dot Product Transformation

Claim: Let $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns. For all $\vec{x}_i, \vec{x}_j \in \mathcal{V}$:

$$\|\mathbf{V}^T\vec{x}_i - \mathbf{V}^T\vec{x}_j\|_2 = \|\vec{x}_i - \vec{x}_j\|_2.$$

$$\|y\|_2 = \sqrt{\sum_{i=1}^{d} y(i)^2}$$

$$\|y\|_2^2 = \sum_{i=1}^{d} y(i)^2 = \langle y, y \rangle$$

$$\left(\sum_{i=1}^{d} y(i) \cdot y(i)\right) = \sum_{i=1}^{d} y(i)^2$$

$$y^T y = \sum_{i=1}^{d} y(i)^2 \quad (1 \times 1)$$

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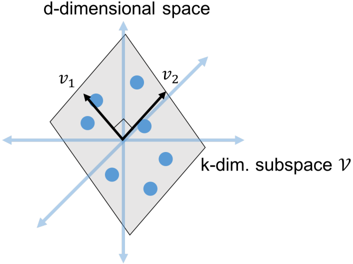

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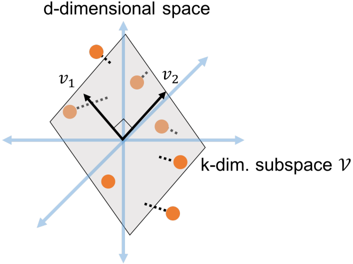

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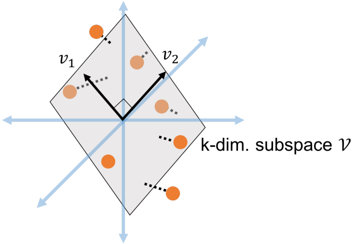

Letting $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$ and $\mathbf{V} \in \mathbb{R}^{d \times k}$ be the matrix with these vectors as its columns, $\mathbf{V}^T \vec{x}_i \in \mathbb{R}^k$ is still a good embedding for $x_i \in \mathbb{R}^d$.

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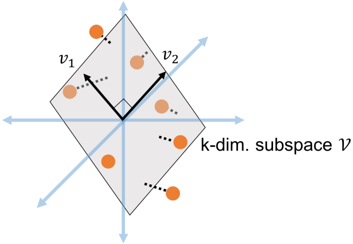

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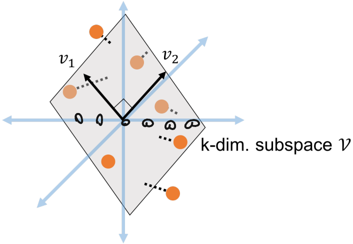

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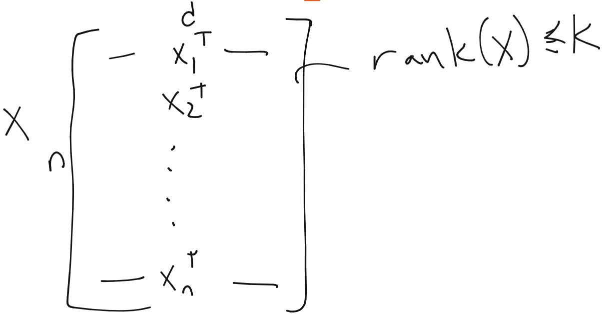
- How do we find $\mathcal{V}$ and $\mathbf{V}$?
- How good is the embedding?

Low-Rank Factorization

Claim: $\vec{x}_1, \ldots, \vec{x}_n$ lie in a k-dimensional subspace $\mathcal{V} \Leftrightarrow$ the data matrix $\mathbf{X} \in \mathbb{R}^{n \times d}$ has rank $\leq k$.



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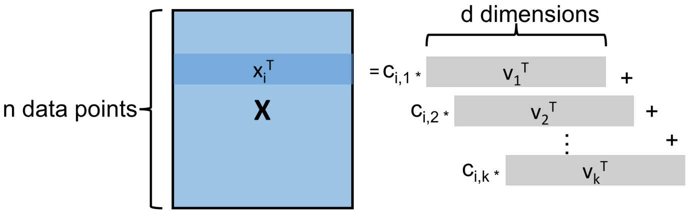
- Letting $\vec{v}_1, \ldots, \vec{v}_k$ be an orthonormal basis for $\mathcal{V}$, can write $\vec{x}_i$ as:

$$\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + c_{i,2} \cdot \vec{v}_2 + \ldots + c_{i,k} \cdot \vec{v}_k.$$



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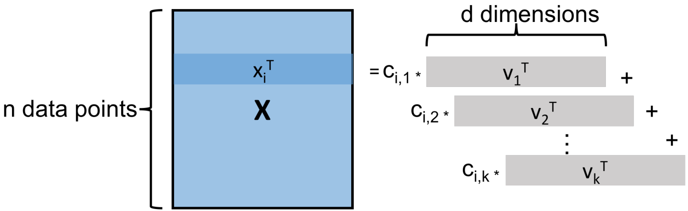
- So $\vec{v}_1, \ldots, \vec{v}_k$ span the rows of $\mathbf{X}$ and thus $\mathrm{rank}(\mathbf{X}) \leq k$.



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- Every data point $\vec{x}_i$ (row of $\mathbf{X}$) can be written as $\vec{x}_i = \mathbf{V}\vec{c}_i = c_{i,1} \cdot \vec{v}_1 + \ldots + c_{i,k} \cdot \vec{v}_k$.

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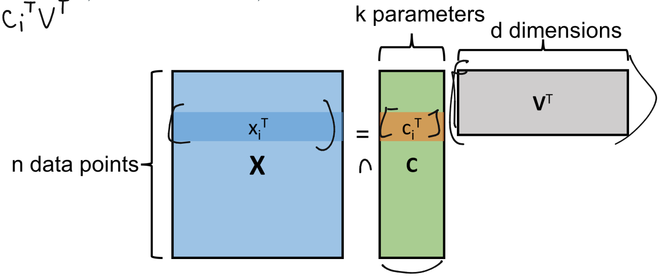
$x_i^T = c_i^T V^T$



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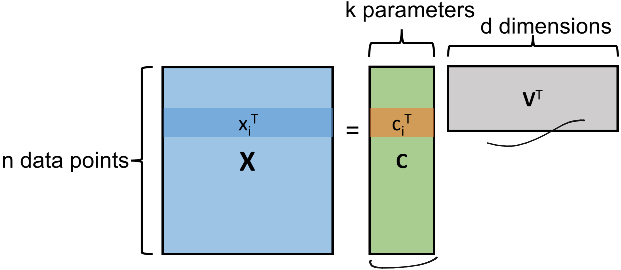
rank(X) = k



- $\mathbf{X}$ can be represented by $(n + d) \cdot k$ parameters vs. $n \cdot d$.

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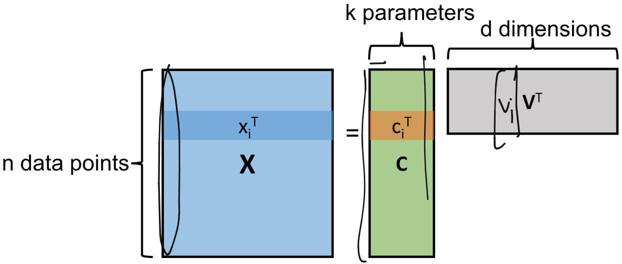

- $\mathbf{X}$ can be represented by $(n + d) \cdot k$ parameters vs. $n \cdot d$.
- The rows of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{V} \implies$ the columns of $\mathbf{X}$ are spanned by k vectors: the columns of $\mathbf{C}$.

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Low-Rank Factorization

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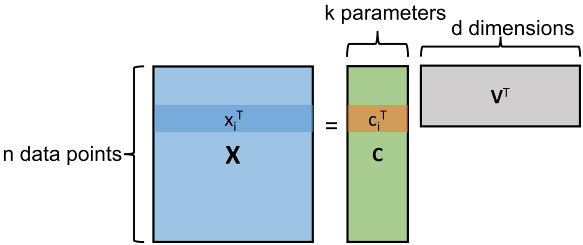

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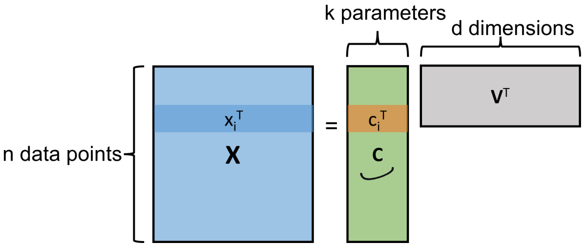

Exercise: What is this coefficient matrix $\mathbf{C}$? **Hint:** Use that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$.

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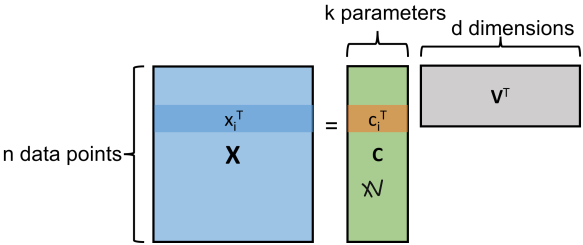
$V^T x_i = c_i$



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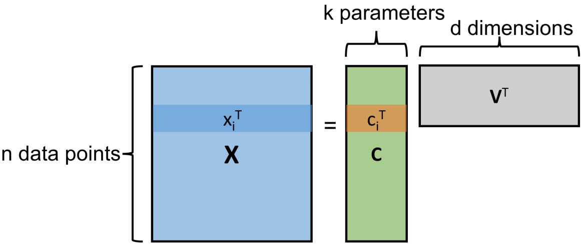

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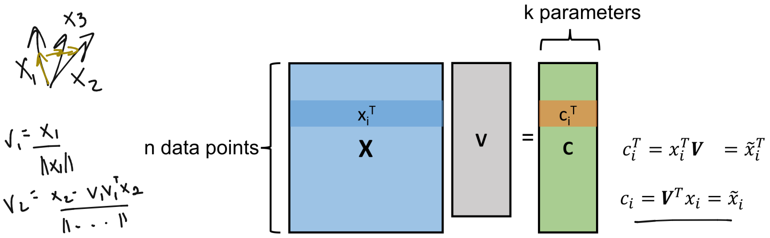

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C = XV

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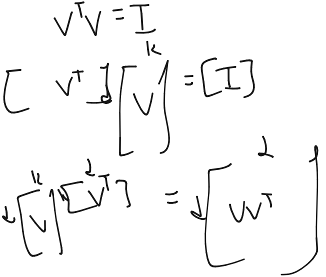

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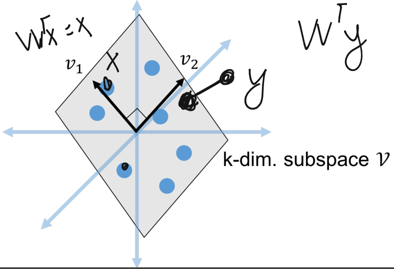

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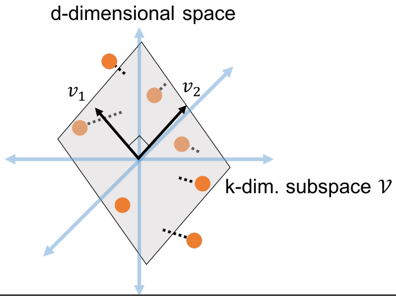

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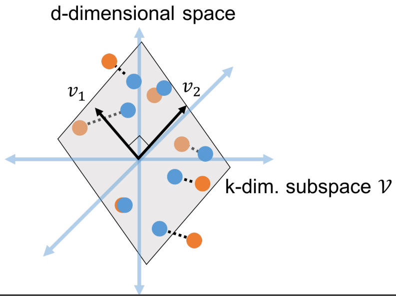

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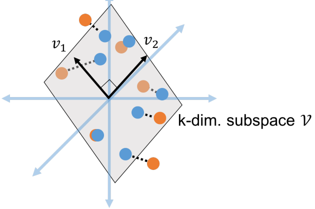

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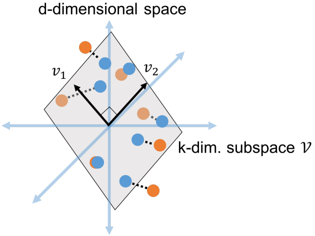

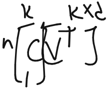

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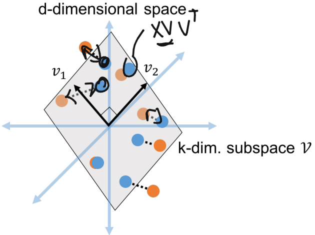

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$$\mathbf{X}\mathbf{V}\mathbf{V}^T = \underset{\mathbf{B} \text{ with rows in } \mathcal{V}}{\arg\min} \|\mathbf{X} - \mathbf{B}\|_F^2 = \sum_{i,j} (\mathbf{X}_{i,j} - \mathbf{B}_{i,j})^2.$$

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Key question is how to find the subspace $\mathcal{V}$ and correspondingly $\mathbf{V}$.

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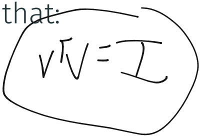
Properties of Projection Matrices

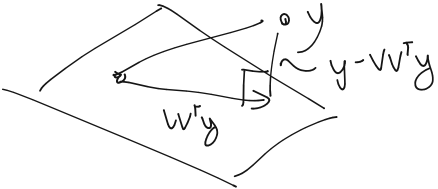
Quick Exercise: Show that $\mathbf{V}\mathbf{V}^T$ is idempotent. I.e., $(\mathbf{V}\mathbf{V}^T)(\mathbf{V}\mathbf{V}^T)\vec{y} = (\mathbf{V}\mathbf{V}^T)\vec{y}$ for any $\vec{y} \in \mathbb{R}^d$.

Why does this make sense intuitively?

Less Quick Exercise: (Pythagorean Theorem) Show that:



$$\|\vec{y}\|_2^2 = \|(\mathbf{V}\mathbf{V}^T)\vec{y}\|_2^2 + \|\vec{y} - (\mathbf{V}\mathbf{V}^T)\vec{y}\|_2^2.$$



A Step Back: Why Low-Rank Approximation?

Question: Why might we expect $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ to lie close to a k-dimensional subspace?

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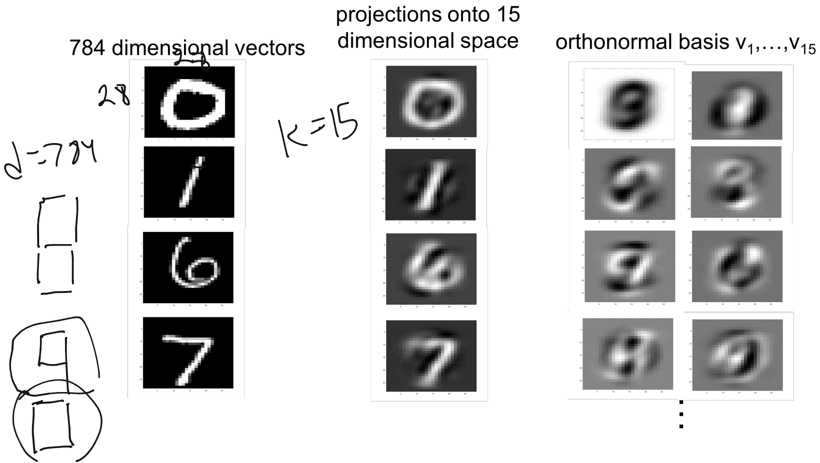

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home 1	2	2	1800	2	200,000	195,000
home 2	4	2.5	2700	1	300,000	310,000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
home n	5	3.5	3600	3	450,000	450,000

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