COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2024.

Lecture 12

Logistics

- Problem Set 2 due tomorrow at 11:59pm.
- Solutions will be released shortly after submission so they can be used to study.
- No quiz this week.
- Midterm review office hours: Tuesday 10/15 1:30-3pm Herter Hall 205. Wednesday 10/16 10am-11:30am Goessmann 151. Thursday in class.
- The midterm will be Thursday 7pm-9pm in ILC N151.

Summary

Last Class:

- Intro to dimensionality reduction.
- Intro to low-distortion embeddings and the Johnson-Lindenstrauss Lemma.

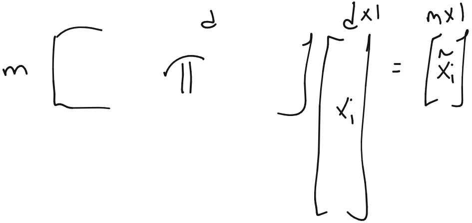
This Class:

- Reduction of JL Lemma to the Distributional JL Lemma.
- Proof the Distributional JL Lemma.
- Example application of JL to clustering.

Johnson-Lindenstrauss Lemma: For any set of points $\vec{x}_1, \ldots, \vec{x}_n \in \mathbb{R}^d$ and $\epsilon > 0$ there exists a linear map $\mathbf{\Pi} : \mathbb{R}^d \to \mathbb{R}^m$ such that $m = O\left(\frac{\log n}{\epsilon^2}\right)$ and letting $\tilde{x}_i = \mathbf{\Pi}\vec{x}_i$:

For all i, j : $(1-\epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\tilde{x}_i - \tilde{x}_j\|_2 \leq (1+\epsilon)\|\vec{x}_i - \vec{x}_j\|_2$.

Further, if $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ has each entry chosen i.i.d. from $\mathcal{N}(0, 1/m)$, it satisfies the guarantee with high probability.

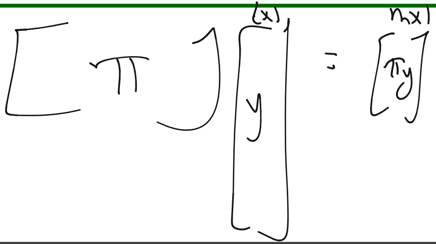


Distributional JL

The Johnson-Lindenstrauss Lemma is a direct consequence of a closely related lemma:

Distributional JL Lemma: Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1-\epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1+\epsilon)\|\vec{y}\|_2$$



$\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection matrix. d: original dimension. m: compressed dimension, ϵ: embedding error, δ: embedding failure prob.

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$$(1-\epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1+\epsilon)\|\vec{y}\|_2$$

Applying a random matrix $\mathbf{\Pi}$ to any vector $\vec{y}$ preserves $\vec{y}$'s norm with high probability.

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Applying a random matrix $\mathbf{\Pi}$ to any vector $\vec{y}$ preserves $\vec{y}$'s norm with high probability.

- Like a low-distortion embedding, but for the length of a compressed vector rather than distances between vectors.
- Can be proven from first principles.

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Distributional JL $\implies$ JL

Distributional JL Lemma $\implies$ JL Lemma: Distributional JL show that a random projection $\mathbf{\Pi}$ preserves the norm of any y. The main JL Lemma says that $\mathbf{\Pi}$ preserves distances between vectors.

$\vec{x}_1, \ldots, \vec{x}_n$: original points, $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_n$: compressed points, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection matrix. d: original dimension. m: compressed dimension, ϵ: embedding error, δ: embedding failure prob.

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Since $\mathbf{\Pi}$ is linear these are the same thing!

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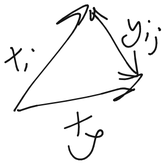
Proof: Given $\vec{x}_1, \ldots, \vec{x}_n$, define $\binom{n}{2}$ vectors $\vec{y}_{ij}$ where $\vec{y}_{ij} = \vec{x}_i - \vec{x}_j$.



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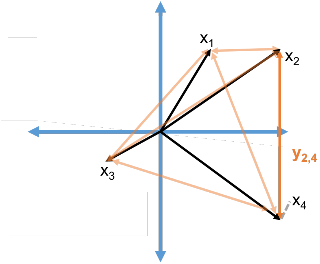

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- If we choose $\mathbf{\Pi}$ with $m = O\left(\frac{\log 1/\delta}{\epsilon^2}\right)$, for each $\vec{y}_{ij}$ with probability $\geq 1 - \delta$ we have:

$$(1-\epsilon)\|\vec{y}_{ij}\|_2 \leq \|\mathbf{\Pi}\vec{y}_{ij}\|_2 \leq (1+\epsilon)\|\vec{y}_{ij}\|_2$$

$x_i - x_j$ $\quad$ $x_i - x_j$ $\quad$ $x_i - x_j$

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Πy_{ij}

$$(1-\epsilon)\|\vec{x}_i - \vec{x}_j\|_2 \leq \|\mathbf{\Pi}(\vec{x}_i - \vec{x}_j)\|_2 \leq (1+\epsilon)\|\vec{x}_i - \vec{x}_j\|_2$$

$\|\Pi x_i - \Pi x_j\|_2$

$\|\tilde{x}_i - \tilde{x}_j\|$

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Claim: If we choose $\mathbf{\Pi}$ with i.i.d. $\mathcal{N}(0, 1/m)$ entries and $m = O\left(\frac{\log(1/\delta')}{\epsilon^2}\right)$, letting $\tilde{\mathbf{x}}_i = \mathbf{\Pi}\vec{x}_i$, for each pair $\vec{x}_i, \vec{x}_j$ with probability $\geq 1 - \delta'$ we have:

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$(1-\delta')^{\binom{n}{2}} \approx 1 - \binom{n}{2}\delta'$

$y_{ij} : x_i - x_j$ $\quad [x_1], [x_2] \ldots [x_n]$

With what probability are all pairwise distances preserved?

Union bound

A_{ij} = event that I don't preserve distance $x_i - x_j$

$1 - \Pr\left(\bigcup_{i,j} A_{ij}\right)$

$\leq \sum_{ij} \Pr(A_{ij}) = \binom{n}{2}\delta'$ (union bound)

$1 - \binom{n}{2}\delta'$

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Apply the claim with $\delta' = \delta / \binom{n}{2}$.

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Yields the JL lemma.

$O\left(\frac{\log n}{\epsilon^2}\right)$

Distributional JL Proof

Distributional JL Lemma: Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1-\epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1+\epsilon)\|\vec{y}\|_2$$

$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection. d: original dim. m: compressed dim, ϵ: error, δ: failure prob.

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m×1 m×d d×1

- Let $\tilde{\mathbf{y}}$ denote $\mathbf{\Pi}\vec{y}$ and let $\mathbf{\Pi}(j)$ denote the j^{th} row of $\mathbf{\Pi}$.

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- Let $\tilde{\mathbf{y}}$ denote $\mathbf{\Pi}\vec{y}$ and let $\mathbf{\Pi}(j)$ denote the j^{th} row of $\mathbf{\Pi}$.
- For any j, $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle$.

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Distributional JL Proof

Distributional JL Lemma: Let $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$. If we set $m = O\left(\frac{\log(1/\delta)}{\epsilon^2}\right)$, then for any $\vec{y} \in \mathbb{R}^d$, with probability $\geq 1 - \delta$

$$(1-\epsilon)\|\vec{y}\|_2 \leq \|\mathbf{\Pi}\vec{y}\|_2 \leq (1+\epsilon)\|\vec{y}\|_2$$

- Let $\tilde{\mathbf{y}}$ denote $\mathbf{\Pi}\vec{y}$ and let $\mathbf{\Pi}(j)$ denote the j^{th} row of $\mathbf{\Pi}$.
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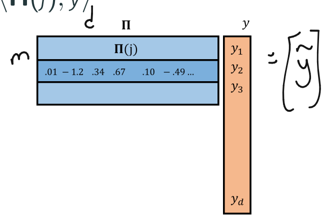

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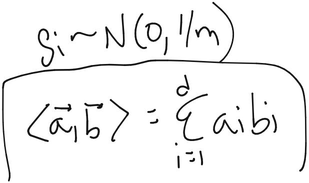

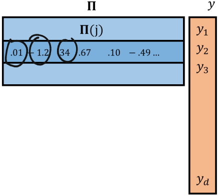

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$N(0, \frac{1}{m})$

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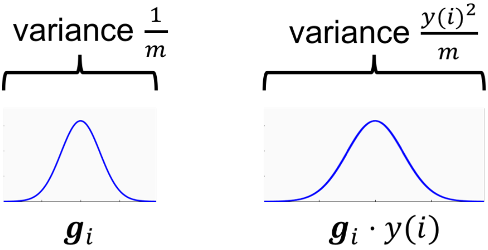

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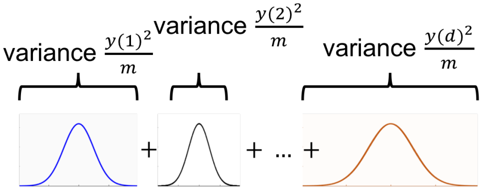

$$\tilde{\mathbf{y}}(j) = [\boldsymbol{g}_1 \cdot y(1) + \boldsymbol{g}_2 \cdot y(2) + \dots + \boldsymbol{g}_n \cdot y(d)]$$

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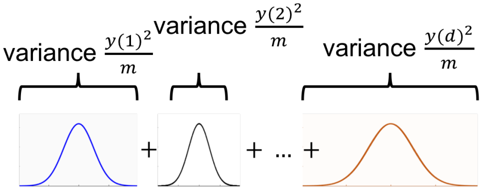

$$\tilde{\mathbf{y}}(j) = [\boldsymbol{g}_1 \cdot y(1) + \boldsymbol{g}_2 \cdot y(2) + \ldots + \boldsymbol{g}_n \cdot y(d)]$$

What is the distribution of $\tilde{\mathbf{y}}(j)$?

> $\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \to \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: compressed dimension, $\mathbf{g}_i$: normally distributed random variable.

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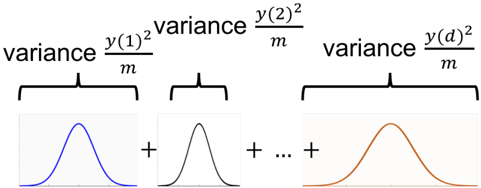

$$\tilde{\mathbf{y}}(j) = [\boldsymbol{g}_1 \cdot y(1) + \boldsymbol{g}_2 \cdot y(2) + \ldots + \boldsymbol{g}_n \cdot y(d)]$$

What is the distribution of $\tilde{\mathbf{y}}(j)$? Also Gaussian!

$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \rightarrow \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: compressed dimension, $\mathbf{g}_i$: normally distributed random variable.

Distributional JL Proof

Letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$, we have $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle$ and:

$$\tilde{\mathbf{y}}(j) = \sum_{i=1}^{d} \mathbf{g}_i \cdot \vec{y}(i) \text{ where } \mathbf{g}_i \cdot \vec{y}(i) \sim \mathcal{N}\left(0, \frac{\vec{y}(i)^2}{m}\right).$$

$$\mathbb{E}[\tilde{y}(j)] = 0$$

$$\mathrm{Var}[\tilde{y}(j)] = \sum_{i=1}^{d} \frac{y(i)^2}{m} = \frac{1}{m}\sum_{i=1}^{d} y(i)^2 = \frac{\|y\|_2^2}{m}$$

$$\tilde{y}(j) \sim N\left(0, \frac{\|y\|_2^2}{m}\right)$$

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Stability of Gaussian Random Variables. For independent $a \sim \mathcal{N}(\mu_1, \sigma_1^2)$ and $b \sim \mathcal{N}(\mu_2, \sigma_2^2)$ we have:

$$a + b \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

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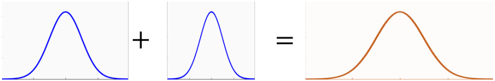

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Thus, $\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \frac{\vec{y}(1)^2}{m} + \frac{\vec{y}(2)^2}{m} + \ldots + \frac{\vec{y}(d)^2}{m})$

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Distributional JL Proof

Letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$, we have $\tilde{\mathbf{y}}(j) = \langle \mathbf{\Pi}(j), \vec{y} \rangle$ and:

$\langle \Pi(j), y \rangle = \tilde{y}$

$$\tilde{\mathbf{y}}(j) = \sum_{i=1}^{d} \mathbf{g}_i \cdot \vec{y}(i) \text{ where } \mathbf{g}_i \cdot \vec{y}(i) \sim \mathcal{N}\left(0, \frac{\vec{y}(i)^2}{m}\right).$$

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Thus, $\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \frac{\|\vec{y}\|_2^2}{m})$ I.e., $\tilde{\mathbf{y}}$ itself is a random Gaussian vector. Rotational invariance of the Gaussian distribution.

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So far: Letting $\mathbf{\Pi} \in \mathbb{R}^{d \times m}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$, for any $\vec{y} \in \mathbb{R}^d$, letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$:

$$\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \|\vec{y}\|_2^2/m).$$

$\|\tilde{y}\|_2 \approx \|y\|_2$

$\vec{y} \in \mathbb{R}^d$: arbitrary vector, $\tilde{\mathbf{y}} \in \mathbb{R}^m$: compressed vector, $\mathbf{\Pi} \in \mathbb{R}^{m \times d}$: random projection mapping $\vec{y} \to \tilde{\mathbf{y}}$. $\mathbf{\Pi}(j)$: j^{th} row of $\mathbf{\Pi}$, d: original dimension. m: compressed dimension, $\mathbf{g}_i$: normally distributed random variable

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$$\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \|\vec{y}\|_2^2/m).$$

$$\mathbb{E}[\tilde{y}(j)] = 0$$

What is $\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2]$?

$$\mathbb{E}\left[\sum_{i=1}^m \tilde{y}(i)^2\right] = \sum_{j=1}^m \mathbb{E}[\tilde{y}(j)^2] = \sum_{j=1}^m \mathrm{Var}(\tilde{y}(j)) = \sum_{j=1}^m \frac{\|y\|_2^2}{m} = \|y\|_2^2$$

$$\mathrm{Var}(\tilde{y}(j)) = \mathbb{E}[\tilde{y}(j)^2] - \mathbb{E}[\tilde{y}(j)]^2 \quad (\mathbb{E}[\tilde{y}(j)] = 0)$$

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What is $\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2]$?

$$\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2] = \mathbb{E}\left[\sum_{j=1}^{m} \tilde{\mathbf{y}}(j)^2\right]$$

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$$\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2] = \mathbb{E}\left[\sum_{j=1}^{m} \tilde{\mathbf{y}}(j)^2\right] = \sum_{j=1}^{m} \mathbb{E}[\tilde{\mathbf{y}}(j)^2]$$

$$= \sum_{j=1}^{m} \frac{\|\vec{y}\|_2^2}{m}$$

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$$= \sum_{j=1}^{m} \frac{\|\vec{y}\|_2^2}{m} = \|\vec{y}\|_2^2$$

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So far: Letting $\mathbf{\Pi} \in \mathbb{R}^{d \times m}$ have each entry chosen i.i.d. as $\mathcal{N}(0, 1/m)$, for any $\vec{y} \in \mathbb{R}^d$, letting $\tilde{\mathbf{y}} = \mathbf{\Pi}\vec{y}$:

$$\tilde{\mathbf{y}}(j) \sim \mathcal{N}(0, \|\vec{y}\|_2^2/m).$$

What is $\mathbb{E}[\|\tilde{\mathbf{y}}\|_2^2]$?

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$$= \sum_{j=1}^m \frac{\|\vec{y}\|_2^2}{m} = \|\vec{y}\|_2^2$$

So $\tilde{\mathbf{y}}$ has the right norm in expectation.

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How is $\|\tilde{\mathbf{y}}\|_2^2$ distributed? Does it concentrate?

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Gaussian

$\|\tilde{\mathbf{y}}\|_2^2 = \sum_{i=1}^m \tilde{\mathbf{y}}(j)^2$ a Chi-Squared random variable with m degrees of freedom (a sum of m squared independent Gaussians)

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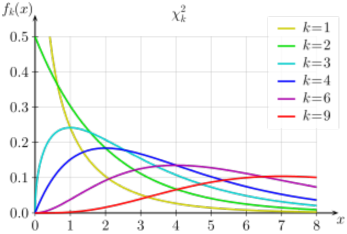

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Lemma: (Chi-Squared Concentration) Letting $\mathbf{Z}$ be a Chi-Squared random variable with m degrees of freedom,

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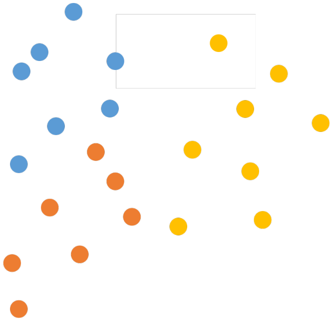
Gives the distributional JL Lemma and thus the classic JL Lemma!

Example Application: k-means clustering

Goal: Separate n points in d dimensional space into k groups.



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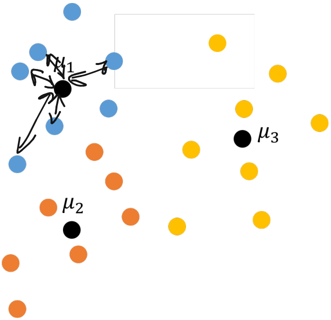

$$\mu_j = \frac{1}{|C_j|} \cdot \sum_{x \in C_j} \vec{x}$$

k-means Objective: $Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x} \in \mathcal{C}_k} \|\vec{x} - \mu_j\|_2^2.$

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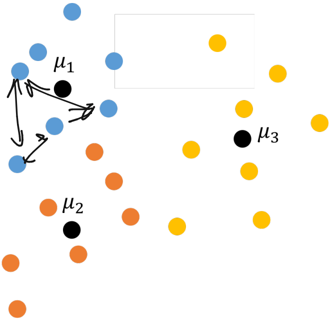

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Write in terms of distances:

$$Cost(\mathcal{C}_1, \ldots, \mathcal{C}_k) = \min_{\mathcal{C}_1, \ldots \mathcal{C}_k} \sum_{j=1}^{k} \sum_{\vec{x}_1, \vec{x}_2 \in \mathcal{C}_k} \|\vec{x}_1 - \vec{x}_2\|_2^2$$

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If we randomly project to $m = O\left(\frac{\log n}{\epsilon^2}\right)$ dimensions, for all pairs $\vec{x}_1, \vec{x}_2$,

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$\geq (1-\varepsilon)\|x_1 - x_2\|_2^2$

$\leq (1+\varepsilon)\|x_1 - x_2\|_2^2$

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Upshot: Can cluster in m dimensional space (much more efficiently) and minimize $\overline{Cost}(\mathcal{C}_1,\ldots,\mathcal{C}_k)$. The optimal set of clusters will have true cost within $1 + c\epsilon$ times the true optimal. Good exercise to prove this.