COMPSCI 514: Algorithms for Data Science

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University of Massachusetts Amherst. Fall 2022.

Lecture 2

Reminder

Reminders:

- Sign up for Piazza.
- Find homework teammates (see Piazza Post) and sign up for Gradescope (code on course website).
- First quiz due Mon 8pm

Overview

Last Class:

- Basic probability review. See course site for links to resources to refresh your probability background.
- Linearity of expectation: $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ always.

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Today:

- Linearity of variance: when does $\text{Var}[X + Y] = \text{Var}[X] + \text{Var}[Y]$?
- Algorithmic applications of linearity of expectation and variance.
- Introduce Markov's inequality a fundamental concentration bound that let us prove that a random variable lies close to its expectation with good probability.
- Learn about random hash functions, which are a key tool in randomized methods for data processing. Probabilistic analysis via linearity of expectation.

Linearity of Variance

$$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$$

Linearity of Variance

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y]$ when X and Y are independent.

uncorrelated

$X = Y$

$\text{Var}(X+Y) = \text{Var}(2X)$

$= 4 \cdot \text{Var}(X) \neq \text{Var}(X) + \text{Var}(Y)$

Linearity of Variance

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Claim 1: (exercise) $\text{Var}[X] = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$\text{Var}(X) = \mathbb{E}((X - \mathbb{E}X)^2)$

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$\sum_s \sum_t s \cdot t \cdot \Pr(X=s \text{ and } Y=t)$

Claim 2: (exercise) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (i.e., X and Y are uncorrelated) when X, Y are independent.

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Together give:

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Together give:

$$\mathrm{Var}[X+Y] = \mathbb{E}[(X+Y)^2] - \mathbb{E}[X+Y]^2$$

$$(\mathbb{E}X + \mathbb{E}Y)^2 = (\mathbb{E}X)^2 + 2\,\mathbb{E}X\,\mathbb{E}Y + (\mathbb{E}Y)^2$$

$$\mathbb{E}[X^2 + 2XY + Y^2]$$

$$\mathbb{E}[X^2] + \mathbb{E}[2XY] + \mathbb{E}[Y^2]$$

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Together give:

$$\begin{aligned}\mathrm{Var}[X + Y] &= \mathbb{E}[(X + Y)^2] - \mathbb{E}[X + Y]^2 \\ &= \underbrace{\mathbb{E}[X^2]} + \underbrace{2\mathbb{E}[XY]} + \underbrace{\mathbb{E}[Y^2]} - (\underbrace{\mathbb{E}[X] + \mathbb{E}[Y]})^2\end{aligned}$$

(linearity of expectation)

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&\qquad \text{(linearity of expectation)} \\
&= \mathbb{E}[X^2] + 2\mathbb{E}[XY] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - 2\mathbb{E}[X] \cdot \mathbb{E}[Y] - \mathbb{E}[Y]^2
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\end{aligned}
$$

$\underbrace{\mathbb{E}[X^2] - \mathbb{E}[X]^2}_{\text{Var}(X)} + \underbrace{\mathbb{E}[Y^2] - \mathbb{E}[Y]^2}_{\text{Var}(Y)}$

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$\mathbb{E}X = 0$ $\mathbb{E}Y = 0$ $\mathbb{E}[XY] = 0$

$X = \pm 1$, $= \pm 2$; $Y = \pm 1$, $Y = \pm 2$ — dependent but uncorrelated

Claim 2: (exercise) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$ (i.e., X and Y are uncorrelated) when X, Y are independent.

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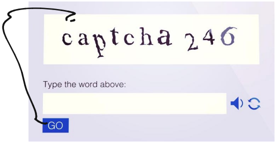
&= \mathbb{E}[X^2] + \mathbb{E}[Y^2] - \mathbb{E}[X]^2 - \mathbb{E}[Y]^2 \\
&= \text{Var}[X] + \text{Var}[Y].
\end{aligned}$$

An Algorithmic Application

You have contracted with a new company to provide CAPTCHAS for your website.



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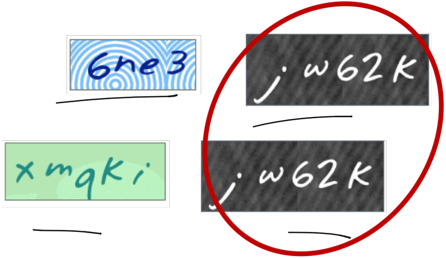
- You want to independently verify this claimed database size.
- You could make test checks until you see $1,000,000$ unique CAPTCHAS: would take $\geq 1,000,000$ checks!

An Algorithmic Application

An Idea: You run some test security checks and see if any duplicate CAPTCHAS show up. If you're seeing duplicates after not too many checks, the database size is probably not too big.



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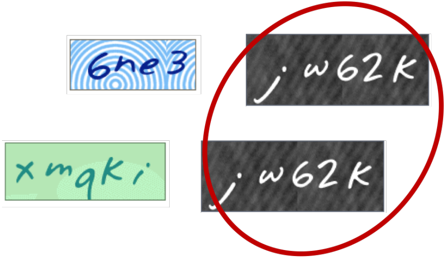

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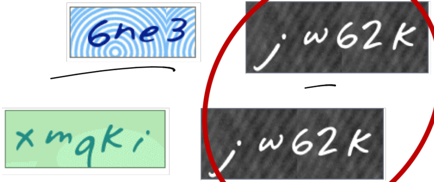

'Mark and recapture' method in ecology.

Think-Pair-Share: If you run m security checks, and there are n unique CAPTCHAS, how many pairwise duplicates do you see in expectation?

If e.g. the same CAPTCHA shows up three times, on your i^{th}, j^{th}, and k^{th} test, this is three duplicates: (i, j), (i, k) and (j, k).

Linearity of Expectation

Let $\mathbf{D}_{i,j} = 1$ if tests i and j give the same CAPTCHA, and 0 otherwise. An indicator random variable.

n: number of CAPTCHAS in database, m: number of random CAPTCHAS drawn to check database size, $\mathbf{D}$: number of pairwise duplicates in m random CAPTCHAS

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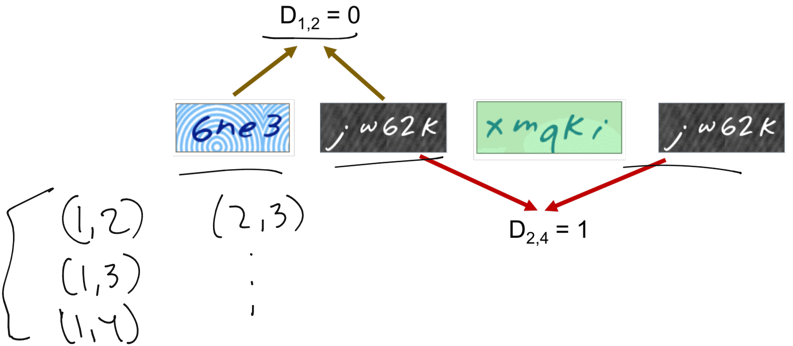

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$$\mathsf{D} = \sum_{i,j \in [m], i<j} \mathsf{D}_{i,j}.$$

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For any pair $i, j \in [m], i < j$: $\mathbb{E}[D_{i,j}] = \Pr[D_{i,j} = 1] = \frac{1}{n}$.

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1/n

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$$\mathbb{E}[\mathsf{D}] = \sum_{i,j \in [m], i<j} \frac{1}{n} = \frac{\binom{m}{2}}{n} = \frac{m(m-1)}{2n}.$$

$\frac{1}{n} \cdot$ # terms in sum $= \binom{m}{2}$

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Note that the $\mathbf{D}_{i,j}$ random variables are not independent!

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Connection to the Birthday Paradox

If there are a 130 people in this room, each whose birthday we assume to be a uniformly random day of the 365 days in the year, how many pairwise duplicate birthdays do we expect there are?

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$$\mathbb{E}[D] = \frac{m(m-1)}{2n} = \frac{130 \cdot 129}{2 \cdot 365} \approx 23.$$

Linearity of Expectation

You take $m = 1000$ samples. If the database size is as claimed ($n = 1,000,000$) then expected number of duplicates is:

$$\mathbb{E}[\mathsf{D}] = \frac{m(m-1)}{2n} = .4995$$

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Concentration Inequalities: Bounds on the probability that a random variable deviates a certain distance from its mean.

- Useful in understanding how statistical tests perform, the behavior of randomized algorithms, the behavior of data drawn from different distributions, etc.

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Useful form: $\Pr[X \geq t \cdot \mathbb{E}[X]] \leq \frac{1}{t}$.

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Useful form: $\Pr[X \geq t \cdot \mathbb{E}[X]] \leq \frac{1}{t}$.

The larger the deviation t, the smaller the probability.

Back to Our Application

Expected number of duplicate CAPTCHAS: 1000 tests

$\mathbb{E}[D] = \frac{m(m-1)}{2n} = .4995.$

You see D = 10 duplicates.

$$\Pr(D \geq 10) \leq \frac{\mathbb{E}[D]}{10} = \frac{.4995}{10} \approx .05$$

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Expected number of duplicate CAPTCHAS:

$\mathbb{E}[\mathrm{D}] = \frac{m(m-1)}{2n} = .4995.$

You see $\mathrm{D} = 10$ duplicates.

Applying Markov's inequality, if the real database size is $n = 1,000,000$ the probability of this happening is:

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This is pretty small – you feel pretty sure the number of unique CAPTCHAS is much less than $1,000,000$. But how can you boost your confidence?

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dict
hash-map

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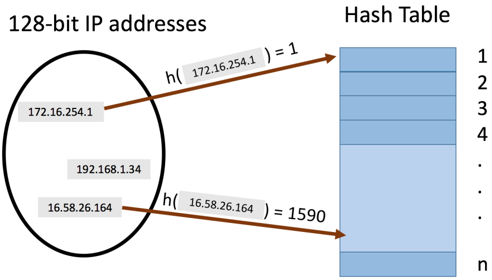
- *Static hashing* since we won't worry about insertion and deletion today.

Hash Tables



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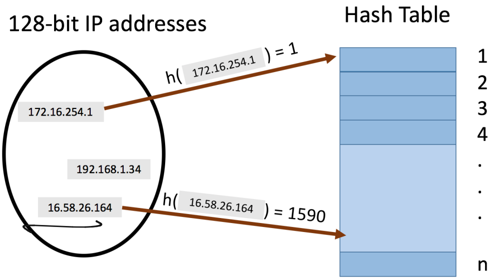

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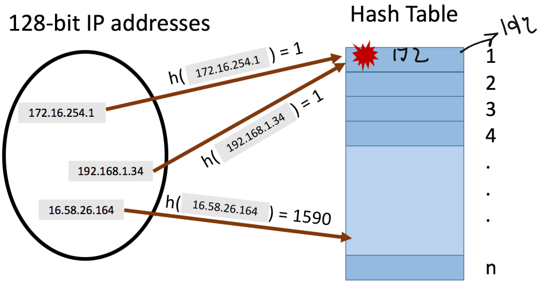
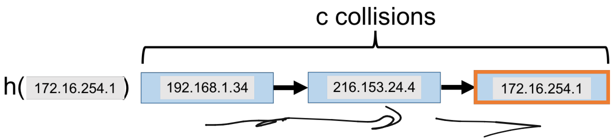


- **hash function** $h : U \to [n]$ maps elements from the universe to indices $1, \cdots, n$ of an array.
- Typically $|U| \gg n$. Many elements map to the same index.
- **Collisions:** when we insert m items into the hash table we may have to store multiple items in the same location (typically as a linked list).

Collisions

Query runtime: $O(c)$ when the maximum number of collisions in a table entry is c (i.e., must traverse a linked list of size c).



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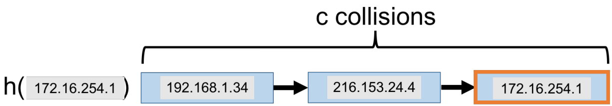

How Can We Bound c?

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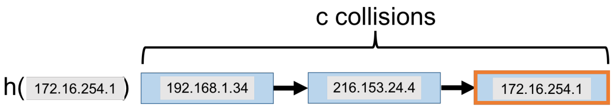

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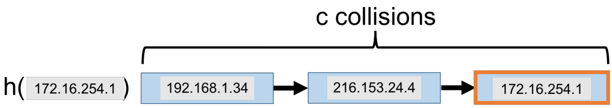

How Can We Bound c?

- In the worst case could have $c = m$ (all items hash to the same location).
- Two approaches: 1) we assume the items inserted are chosen randomly from the universe U or 2) we assume the hash function is random.

Random Hash Function

Let $\mathbf{h} : U \to [n]$ be a fully random hash function.

- I.e., for $x \in U$, $\Pr(\mathbf{h}(x) = i) = \frac{1}{n}$ for all $i = 1, \ldots, n$ and $\mathbf{h}(x), \mathbf{h}(y)$ are independent for any two items $x \neq y$.

$h(x) = h(x)$

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- **Caveat 1:** It is *very expensive* to represent and compute such a random function. We will later see how a hash function computable in $O(1)$ time function can be used instead.
- **Caveat 2:** In practice, often suffices to use hash functions like MD5, SHA-2, etc. that 'look random enough'.

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Think-Pair-Share: Assuming we insert m elements into a hash table of size n using a fully random hash function, what is the expected total number of pairwise collisions?