COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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University of Massachusetts Amherst. Fall 2021.
Lecture 6

- Problem Set 1 is due this Friday at 8pm in Gradescope.
- My office hours have moved to Thursday 5-6pm on Zoom.

Last Class:

- Exponential concentration bounds – Bernstein and Chernoff
- Connection to the central limit theorem

This Class:

- Bloom filters: random hashing to maintain a large set in small space.
- Possibly start on distinct items counting

Want to store a set S of items from a massive universe of possible items (e.g., images, text documents, IP addresses).

Goal: support *insert*(x) to add x to the set and *query*(x) to check if x is in the set. Both in $O(1)$ time. What data structure solves this problem?

- Allow small probability $\delta > 0$ of false positives. I.e., for any x,

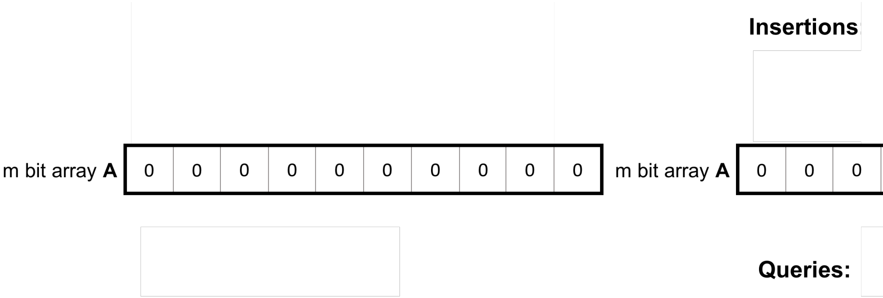
$$\Pr(query(x) = 1 \text{ and } x \notin S) \leq \delta.$$

Solution: Bloom filters (repeated random hashing). Will use much less space than a hash table.

BLOOM FILTERS

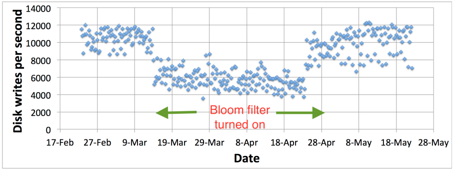
Chose k independent random hash functions $\mathbf{h}_1, \ldots, \mathbf{h}_k$ mapping the universe of elements $U \to [m]$.

- Maintain an array A containing m bits, all initially 0.
- *insert*(x): set all bits $A[\mathbf{h}_1(x)] = \ldots = A[\mathbf{h}_k(x)] := 1$.
- *query*(x): return 1 only if $A[\mathbf{h}_1(x)] = \ldots = A[\mathbf{h}_k(x)] = 1$.



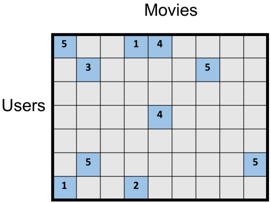
No false negatives. False positives more likely with more insertions.

Akamai (Boston-based company serving 15 – 30% of all web traffic) applies bloom filters to prevent caching of 'one-hit-wonders' – pages only visited once fill over 75% of cache.



- When url x comes in, if $query(x) = 1$, cache the page at x. If not, run $insert(x)$ so that if it comes in again, it will be cached.
- **False positive:** A new url (possible one-hit-wonder) is cached. If the bloom filter has a false positive rate of $\delta = .05$, the number of cached one-hit-wonders will be reduced by at least 95%.

Distributed database systems, including Google Bigtable, Apache HBase, Apache Cassandra, and PostgreSQL use bloom filters to prevent expensive lookups of non-existent data.



- When a new rating is inserted for $(user_x, movie_y)$, add $(user_x, movie_y)$ to a bloom filter.
- Before reading $(user_x, movie_y)$ (possibly requiring an out of memory access), check the bloom filter, which is stored in memory.
- **False positive:** A read is made to a possibly empty cell. A $\delta = .05$ false positive rate gives a 95% reduction in these empty reads.

- Database Joins: Quickly eliminate most keys in one column that don't correspond to keys in another.
- Recommendation systems: Bloom filters are used to prevent showing users the same recommendations twice.
- Spam/Fraud Detection:
  - Bit.ly and Google Chrome use bloom filters to quickly check if a url maps to a flagged site and prevent a user from following it.
  - Can be used to detect repeat clicks on the same ad from a single IP-address, which may be the result of fraud.
- Digital Currency: Some Bitcoin clients use bloom filters to quickly pare down the full transaction log to transactions involving bitcoin addresses that are relevant to them (SPV: simplified payment verification).

For a bloom filter with m bits and k hash functions, the insertion and query time is $O(k)$. How does the false positive rate δ depend on m, k, and the number of items inserted?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0? $n \times k$ total hashes must not hit bit i.

$$\begin{aligned}
\Pr(A[i] = 0) &= \Pr\Big(\mathbf{h}_1(x_1) \neq i \cap \ldots \cap \mathbf{h}_k(x_k) \neq i \\
&\qquad \cap\, \mathbf{h}_1(x_2) \neq i \ldots \cap \mathbf{h}_k(x_2) \neq i \cap \ldots\Big) \\
&= \underbrace{\Pr\left(\mathbf{h}_1(x_1) \neq i\right) \times \ldots \times \Pr\left(\mathbf{h}_k(x_1) \neq i\right) \times \Pr\left(\mathbf{h}_1(x_2) \neq i\right) \ldots}_{k \cdot n \text{ events each occuring with probability } 1 - 1/m} \\
&= \left(1 - \frac{1}{m}\right)^{kn}
\end{aligned}$$

How does the false positive rate δ depend on m, k, and the number of items inserted?

Step 1: What is the probability that after inserting n elements, the i^{th} bit of the array A is still 0?

$$\Pr(A[i]=0) = \left(1-\frac{1}{m}\right)^{kn} \approx e^{-\frac{kn}{m}}$$

Step 2: What is the probability that querying a new item w gives a false positive?

$$\begin{aligned}\Pr\left(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1\right) &= \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1) \\ &= \left(1-e^{-\frac{kn}{m}}\right)^k\end{aligned}$$

Actually Incorrect! Dependent events.

n: total number items in filter, m: number of bits in filter, k: number of random hash functions, $\mathbf{h}_1, \ldots \mathbf{h}_k$: hash functions, A: bit array, δ: false positive rate.

Step 1: To avoid dependence issues, condition on the event that the A has t zeros in it after n insertions, for some $t \leq m$. For a non-inserted element w, after conditioning on this event we correctly have:

$$\begin{aligned}\Pr(A[\mathbf{h}_1(w)] = \ldots = A[\mathbf{h}_k(w)] = 1)\\ &= \Pr(A[\mathbf{h}_1(w)] = 1) \times \ldots \times \Pr(A[\mathbf{h}_k(w)] = 1).\end{aligned}$$

I.e., the events $A[\mathbf{h}_1(w)] = 1$,..., $A[\mathbf{h}_k(w)] = 1$ are independent conditioned on the number of bits set in A. Why?

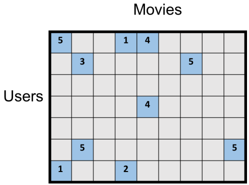
- Conditioned on this event, for any j, since $\mathbf{h}_j$ is a fully random hash function, $\Pr(A[\mathbf{h}_j(w)] = 1) = 1 - \frac{t}{m}$.
- Thus conditioned on this event, the false positive rate is $\left(1 - \frac{t}{m}\right)^k$.
- It remains to show that $\frac{t}{m} \approx e^{-\frac{kn}{m}}$ with high probability. We already have that $\mathbb{E}[\frac{t}{m}] = \frac{1}{m}\sum_{i=1}^{m} \Pr(A[i] = 0) \approx e^{-\frac{kn}{m}}$.

Need to show that the number of zeros t in A after n insertions is bounded by $O\left(e^{-\frac{kn}{m}}\right)$ with high probability.

Can apply Theorem 2 of: `http://cglab.ca/~morin/publications/ds/bloom-submitted.pdf`

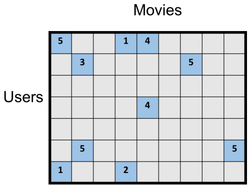
FALSE POSITIVE RATE

False Positive Rate: with m bits of storage, k hash functions, and n items inserted $\delta \approx \left(1 - e^{\frac{-kn}{m}}\right)^k$.



- We have 100 million users and $10,000$ movies. On average each user has rated only 10 movies so of these 10^{12} possible (user,movie) pairs, only $10 * 100,000,000 = 10^9 = n$ (user,movie) pairs have non-empty entries in our table.
- We allocate $m = 8n = 8 \times 10^9$ bits for a Bloom filter (1 GB). How should we set k to minimize the number of false positives?

False Positive Rate: with m bits of storage, k hash functions, and n items inserted $\delta \approx \left(1 - e^{\frac{-kn}{m}}\right)^k$.



- $n = 10^9 = n$ (user,movie) pairs with non-empty entries in our table.
- $m = 8n = 8 \times 10^9$ bits for a Bloom filter (1 GB).
- Set $k = \ln 2 \cdot \frac{m}{n} = 5.54 \approx 6$.
- False positive rate is $\approx \left(1 - e^{-k \cdot \frac{n}{m}}\right)^k \approx \frac{1}{2^k} \approx \frac{1}{2^{5.54}} = .021$.

An observation about Bloom filter space complexity:

$$\text{False Positive Rate: } \delta \approx \left(1 - e^{-\frac{kn}{m}}\right)^k.$$

For an m-bit bloom filter holding n items, optimal number of hash functions k is: $k = \ln 2 \cdot \frac{m}{n}$.

Think Pair Share: If we want a false positive rate $< \frac{1}{2}$ how big does m need to be in comparison to n?

$$m = O(\log n),\ m = O(\sqrt{n}),\ m = O(n),\ m = O(n^2)?$$

If $m = \frac{n}{\ln 2}$, optimal $k = 1$, and failure rate is:

$$\delta = \left(1 - e^{-\frac{n/\ln 2}{n}}\right)^1 = \left(1 - \frac{1}{2}\right)^1 = \frac{1}{2}.$$

I.e., storing n items in a bloom filter requires $O(n)$ space. So what's the point? Truly $O(n)$ bits, rather than $O(n \cdot \text{item size})$.

Questions on Bloom Filters?