COMPSCI 514: ALGORITHMS FOR DATA SCIENCE

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Lecture 5

LOGISTICS

- Problem Set 1 was released on Ttuesday and is next Friday 9/24 at 8pm in Gradescope. Get started thinking over the problems early if you can.
- See Piazza for a poll about potentially moving my office hours time.

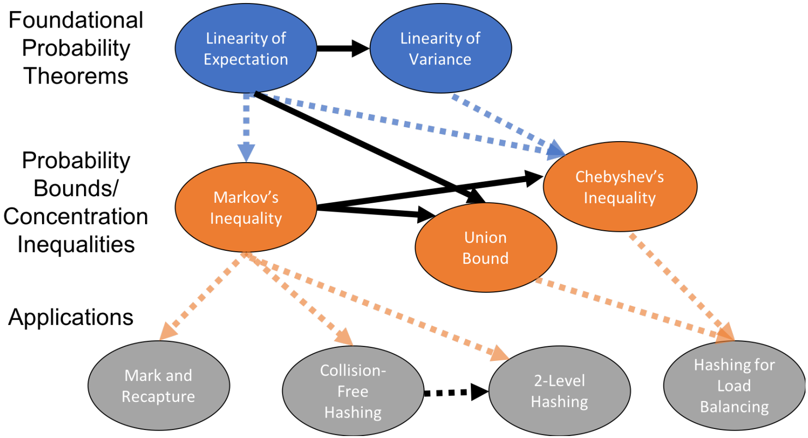
Last Class: Concentration bounds beyond Markov's inequality

- Chebyshev's inequality and the law of large numbers.

This Time:

- Exponential concentration bounds and the central limit theorem.
- Bloom Filters – More efficient 'approximate' hash tables.

CONCEPT MAP


Foundational Probability Theorems
Probability Bounds/ Concentration Inequalities
Applications
Linearity of Expectation
Linearity of Variance
Markov's Inequality
Chebyshev's Inequality
Union Bound
Mark and Recapture
Collision-Free Hashing
2-Level Hashing
Hashing for Load Balancing

We flip $n = 100$ independent coins, each are heads with probability 1/2 and tails with probability 1/2. Let $\mathbf{H}$ be the number of heads.

$$\mathbb{E}[\mathbf{H}] = \frac{n}{2} = 50 \text{ and } \mathrm{Var}[\mathbf{H}] = \frac{n}{4} = 25$$

Markov's:	Chebyshev's:	In Reality:
$\Pr(\mathbf{H} \geq 60) \leq .833$	$\Pr(\mathbf{H} \geq 60) \leq .25$	$\Pr(\mathbf{H} \geq 60) = 0.0284$
$\Pr(\mathbf{H} \geq 70) \leq .714$	$\Pr(\mathbf{H} \geq 70) \leq .0625$	$\Pr(\mathbf{H} \geq 70) = .000039$
$\Pr(\mathbf{H} \geq 80) \leq .625$	$\Pr(\mathbf{H} \geq 80) \leq .0278$	$\Pr(\mathbf{H} \geq 80) < 10^{-9}$

$\mathbf{H}$ has a simple Binomial distribution, so can compute these probabilities exactly.

To be fair.... Markov and Chebyshev's inequalities apply much more generally than to Binomial random variables like coin flips.

Can we obtain tighter concentration bounds that still apply to very general distributions?

- Markov's: $\Pr(X \geq t) \leq \frac{\mathbb{E}[X]}{t}$. First Moment.
- Chebyshev's: $\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr(|X - \mathbb{E}[X]|^2 \geq t^2) \leq \frac{\text{Var}[X]}{t^2}$. Second Moment.
- What if we just apply Markov's inequality to even higher moments?

Consider any random variable X:

$$\Pr(|X - \mathbb{E}[X]| \geq t) = \Pr\left((X - \mathbb{E}[X])^4 \geq t^4\right) \leq \frac{\mathbb{E}\left[(X - \mathbb{E}[X])^4\right]}{t^4}.$$

Application to Coin Flips: Recall: $n = 100$ independent fair coins, H is the number of heads.

- Bound the fourth moment:

$$\mathbb{E}\left[(H - \mathbb{E}[H])^4\right] = \mathbb{E}\left[\left(\sum_{i=1}^{100} H_i - 50\right)^4\right] = \sum_{i,j,k,\ell} c_{ijk\ell}\mathbb{E}[H_i H_j H_k H_\ell] = 1862.5$$

  where $H_i = 1$ if coin flip i is heads and 0 otherwise. Then apply some messy calculations...

- Apply Fourth Moment Bound: $\Pr\left(|H - \mathbb{E}[H]| \geq t\right) \leq \frac{1862.5}{t^4}$.

Chebyshev's:	4^{th} Moment:	In Reality:
$\Pr(\mathbf{H} \geq 60) \leq .25$	$\Pr(\mathbf{H} \geq 60) \leq .186$	$\Pr(\mathbf{H} \geq 60) = 0.0284$
$\Pr(\mathbf{H} \geq 70) \leq .0625$	$\Pr(\mathbf{H} \geq 70) \leq .0116$	$\Pr(\mathbf{H} \geq 70) = .000039$
$\Pr(\mathbf{H} \geq 80) \leq .04$	$\Pr(\mathbf{H} \geq 80) \leq .0023$	$\Pr(\mathbf{H} \geq 80) < 10^{-9}$

Can we just keep applying Markov's inequality to higher and higher moments and getting tighter bounds?

- Yes! To a point.
- In fact – don't need to just apply Markov's to $|\mathbf{X} - \mathbb{E}[\mathbf{X}]|^k$ for some k. Can apply to any monotonic function $f(|\mathbf{X} - \mathbb{E}[\mathbf{X}]|)$.
- Why monotonic? $\Pr(|\mathbf{X} - \mathbb{E}[\mathbf{X}]| > t) = \Pr(f(|\mathbf{X} - \mathbb{E}[\mathbf{X}]|) > f(t))$.

$\mathbf{H}$: total number heads in 100 random coin flips. $\mathbb{E}[\mathbf{H}] = 50$.

Moment Generating Function: Consider for any $t > 0$:

$$M_t(X) = e^{t\cdot(X-\mathbb{E}[X])} = \sum_{k=0}^{\infty} \frac{t^k (X - \mathbb{E}[X])^k}{k!}$$

- $M_t(X)$ is monotonic for any $t > 0$.
- Weighted sum of all moments, with t controlling how slowly the weights fall off (larger t = slower falloff).
- Choosing t appropriately lets one prove a number of very powerful exponential concentration bounds (exponential tail bounds).
- Chernoff bound, Bernstein inequalities, Hoeffding's inequality, Azuma's inequality, Berry-Esseen theorem, etc.
- We will not cover the proofs in the this class, but you will do one on the first problem set.

Bernstein Inequality: Consider independent random variables $X_1, \ldots, X_n$ all falling in $[-M, M]$[-1,1]. Let $\mu = \mathbb{E}[\sum_{i=1}^n X_i]$ and $\sigma^2 = \text{Var}[\sum_{i=1}^n X_i] = \sum_{i=1}^n \text{Var}[X_i]$. For any $t \geq 0$$s \geq 0$:

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sigma^2 + \frac{4}{3}Mt}\right).$$

$$\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq s\sigma\right) \leq 2\exp\left(-\frac{s^2}{4}\right).$$

Assume that $M = 1$ and plug in $t = s \cdot \sigma$ for $s \leq \sigma$.

Compare to Chebyshev's: $\Pr\left(\left|\sum_{i=1}^n X_i - \mu\right| \geq s\sigma\right) \leq \frac{1}{s^2}$.

- An exponentially stronger dependence on s!

Consider again bounding the number of heads H in $n = 100$ independent coin flips.

Chebyshev's:	Bernstein:	In Reality:
$\Pr(H \geq 60) \leq .25$	$\Pr(H \geq 60) \leq .15$	$\Pr(H \geq 60) = 0.0284$
$\Pr(H \geq 70) \leq .0625$	$\Pr(H \geq 70) \leq .00086$	$\Pr(H \geq 70) = .000039$
$\Pr(H \geq 80) \leq .04$	$\Pr(H \geq 80) \leq 3^{-7}$	$\Pr(H \geq 80) < 10^{-9}$

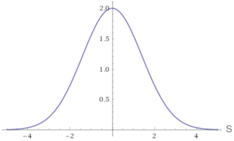
Getting much closer to the true probability.

H: total number heads in 100 random coin flips. $\mathbb{E}[H] = 50$.

Bernstein Inequality (Simplified): Consider independent random variables $X_1, \ldots, X_n$ falling in [-1,1]. Let $\mu = \mathbb{E}[\sum X_i]$, $\sigma^2 = \text{Var}[\sum X_i]$, and $s \leq \sigma$. Then:

$$\Pr\left(\left|\sum_{i=1}^{n} X_i - \mu\right| \geq s\sigma\right) \leq 2\exp\left(-\frac{s^2}{4}\right).$$

Can plot this bound for different s:



Looks a lot like a Gaussian (normal) distribution.

$$\mathcal{N}(0, \sigma^2) \text{ has density } p(s\sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{s^2}{2}}.$$

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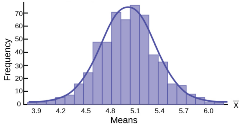
Exercise: Using this can show that for $X \sim \mathcal{N}(0,\sigma^2)$: for any $s \geq 0$,

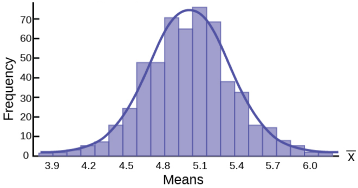
$$\Pr\left(|X| \geq s \cdot \sigma\right) \leq 2e^{-\frac{s^2}{2}}.$$

Essentially the same bound that Bernstein's inequality gives!

Central Limit Theorem Interpretation: Bernstein's inequality gives a quantitative version of the CLT. The distribution of the sum of *bounded* independent random variables can be upper bounded with a Gaussian (normal) distribution.



Stronger Central Limit Theorem: The distribution of the sum of *n bounded* independent random variables converges to a Gaussian (normal) distribution as *n* goes to infinity.



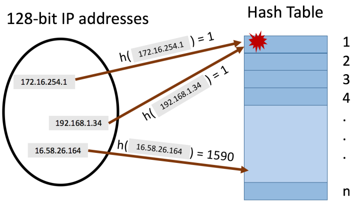
- Why is the Gaussian distribution is so important in statistics, science, ML, etc.?
- Many random variables can be approximated as the sum of a large number of small and roughly independent random effects. Thus, their distribution looks Gaussian by CLT.

A useful variation of the Bernstein inequality for binary (indicator) random variables is:

Chernoff Bound (simplified version): Consider independent random variables $X_1, \ldots, X_n$ taking values in $\{0, 1\}$. Let $\mu = \mathbb{E}[\sum_{i=1}^{n} X_i]$. For any $\delta \geq 0$

$$\Pr\left(\left|\sum_{i=1}^{n} X_i - \mu\right| \geq \delta\mu\right) \leq 2\exp\left(-\frac{\delta^2\mu}{2+\delta}\right).$$

As δ gets larger and larger, the bound falls of exponentially fast.



We hash m values $x_1, \ldots, x_m$ using a random hash function into a table with $n = m$ entries.

- I.e., for all $j \in [m]$ and $i \in [n]$, $\Pr(\mathbf{h}(x) = i) = \frac{1}{m}$ and hash values are chosen independently.

What will be the maximum number of items hashed into the same location?

Let S_i be the number of items hashed into position i and $S_{i,j}$ be 1 if x_j is hashed into bucket i ($h(x_j) = i$) and 0 otherwise.

$$\mathbb{E}[S_i] = \sum_{j=1}^{m} \mathbb{E}[S_{i,j}] = m \cdot \frac{1}{m} = 1 = \mu.$$

By the Chernoff Bound: for any $\delta \geq 0$,

$$\Pr(S_i \geq 1 + \delta) \leq \Pr\left(\left|\sum_{i=1}^{n} S_{i,j} - 1\right| \geq \delta \cdot \mu\right) \leq 2\exp\left(-\frac{\delta^2}{2+\delta}\right)$$

m: total number of items hashed and size of hash table. $x_1, \ldots, x_m$: the items.
h: random hash function mapping $x_1, \ldots, x_m \to [m]$.

$$\Pr(\mathbf{S}_i \geq 1+\delta) \leq \Pr\left(\left|\sum_{i=1}^{n} \mathbf{S}_{i,j} - 1\right| \geq \delta\right) \leq 2\exp\left(-\frac{\delta^2}{2+\delta}\right).$$

Set $\delta = 20 \log m$. Gives:

$$\Pr(\mathbf{S}_i \geq 20\log m + 1) \leq 2\exp\left(-\frac{(20\log m)^2}{2+20\log m}\right) \leq \exp(-18\log m) \leq \frac{2}{m^{18}}.$$

Apply Union Bound:

$$\Pr(\max_{i\in[m]} \mathbf{S}_i \geq 20\log m + 1) = \Pr\left(\bigcup_{i=1}^{m} (\mathbf{S}_i \geq 20\log m + 1)\right)$$

$$\leq \sum_{i=1}^{m} \Pr(\mathbf{S}_i \geq 20\log m + 1) \leq m \cdot \frac{2}{m^{18}} = \frac{2}{m^{17}}.$$

m: total number of items hashed and size of hash table. $\mathbf{S}_i$: number of items hashed to bucket i. $\mathbf{S}_{i,j}$: indicator if x_j is hashed to bucket i. δ: any value ≥ 0.

Upshot: If we randomly hash m items into a hash table with m entries the maximum load per bucket is $O(\log m)$ with very high probability.

- So, even with a simple linked list to store the items in each bucket, worst case query time is $O(\log m)$.
- Using Chebyshev's inequality could only show the maximum load is bounded by $O(\sqrt{m})$ with good probability (good exercise).
- The Chebyshev bound holds even with a pairwise independent hash function. The stronger Chernoff-based bound can be shown to hold with a *k-wise independent hash function* for $k = O(\log m)$.

Questions on Exponential Concentration Bounds?

This conclude the probability foundations part of the course –
on to algorithms.